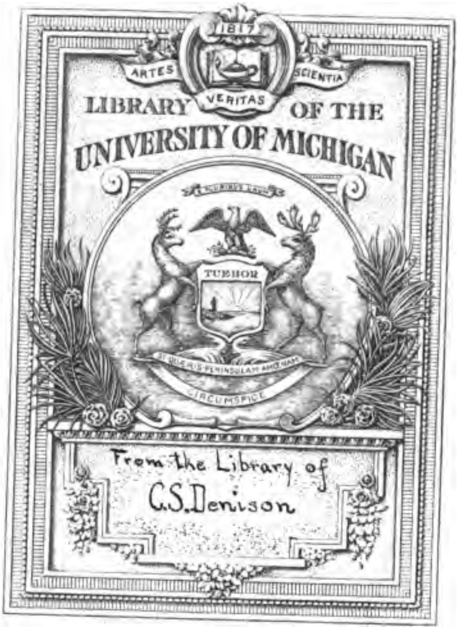
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A

TREATISE

ON

THE DOCTRINE

OF

NUMERICAL SERIES,

BOTH,

ASCENDING AND DESCENDING:

ALSO THE

BINOMIAL THEOREM,

WITH

INTEGER AND FRACTIONAL EXPONENTS.

BY ALONZO JACKMAN, M. A.
Professor of Mathematics in Norwich University.

$$(P+PQ)^{\frac{m}{n}}=P^{\frac{m}{n}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ+\frac{m-2n}{3n}CQ+\frac{m-3n}{4n}DQ+$$

CLAREMONT, N. H.
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1846.

PREFACE.

The substance of the following pages was written about three years since, while I was connected with the New-England Seminary, at Windsor, Vt. My main object was to supply what, in my opinion, is a deficiency relative to the discussion of *Binomial Expansions*, and *Method of Differences*, in President Day's excellent treatise on Algebra, which work was used for a text-book in the Seminary. About two years since, I became connected with the Norwich University, in consequence of which the publication of my manuscript was relinquished. Having, however, of late, been repeatedly solicited by the members of my classes, to publish my method of discoursing on the subjects previously alluded to, I now, in compliance with their wishes, present my old manuscript for publication.

Although I may be mistaken as to the fact, yet, I am not aware that this subject has heretofore been presented in a manner similar to the one now adopted; nor has even a sentence been borrowed from any author;—the whole is my own whether *good* or *bad*.

That this subject has been discussed in the ablest manner, is not pretended. A thorough review, in recitation, will no doubt suggest many important alterations. My chief aim has been to convey information to those who seek for the *why* and the *how* in what they study.

A few errors, which were not noticed in time to be corrected in the text, are arranged under the heading, ERRATA, on the last page of the work. Should other errors unmask themselves to the critic's taste he will please remember, that, were *perfection* required before *presentation*, no human being would receive any thing from another, in this state of existence. With these few remarks, I submit my work to a candid public.

ALONZO JACKMAN.

Norwich University, March 21, 1846.

THE DOCTRINE

OF

NUMERICAL SERIES.

INTRODUCTION.

DEFINITIONS AND PRINCIPLES.

Article 1. The term *Series*, when applied to numerals, denotes a rank of terms, which follow each other in succession, and which are respectively regulated, in their numerical values, according to some *given law*. And as there may be an infinite variety of *laws*, so there may also arise an infinite variety of series.

2. The terms of a *series* may succeed each other, either by *increasing* in their numerical value, or by *decreasing:* in the former case, the terms constitute an *increasing series;* in the latter, a *decreasing series.*

3. When, by a *given law*, one series of terms produces a second series; the second, a third; the third, a fourth, and so on, to an indefinite extent; each series is denominated an *order of series;* and the several orders, collectively, are termed a *system of series.*

That series, from which the several *orders* are derived, is the *basis* of the *system*, and it is also termed the *first order* in the system; the next succeeding order is the *second* in the system; the next, the *third*, and so on.

An *ascending system of series* is one, in which the successive *orders* increase in their numerical values; but when they *decrease*, it is a *descending system of series.*

4. The *law* by which the successive *orders* of a system of series is derived, may be varied infinitely; hence, on the same *basis*, (Art. 3,) an infinite number of different systems may be

formed, and evidently (from Art. 1,) there may be an infinite number of bases, on each of which, as has just been said, an infinite number of systems may arise.

NOTE.—From the foregoing, it is evident, that the subject of *Series* opens before us a vast field for investigation; but our present inquiry will be confined to those series, which may arise under one or two general *laws*. In the following discussion, the reader is supposed to be acquainted with the general principles of Algebra.

NOTATION.

5. In the following discussion, the several terms of a series (Art. 1,) are supposed to be arranged in the order of their succession, on a line from left to right, forming a *horizontal rank*.

6. The successive terms of a series are numbered from left to right; that is, the *first* term is that on the extreme left of the rank, the *second* is next on its right, and so on to the last on the extreme right.

7. When several terms are mentioned collectively, as *the sum of any number of terms*, &c., it is to be understood that they are the consecutive terms on the left of the rank, including the first term. (Arts. 5, 6.)

8. In algebraic investigations, a *numerical value* is frequently represented by a *letter*, which is designed to point out a *particular thing*, (among several of the same kind,) to which the attention is to be directed; yet, in so general a manner, that the truth thus affirmed of it, shall be equally applicable to every individual under the *assumed condition*. Thus, we say of the terms of a series; the 1st, 2d, 3d, 4th, and generally, the nth term. Also, we say; the sum of 2, 3, 4, 5, or of 6 terms, and generally, the sum of n terms.

This mode of expressing a general truth, in the course of a demonstration, is very convenient, as it allows the writer to confine his remarks to a particular thing, (a numerical letter,) which, with all that is affirmed of it, the reader may successively apply to each individual thing under the general hypothesis.

In the same manner that n is used for numeral, we may use the *sum*, or the *difference* of letters: thus, we may say, the $(r-m+t)$th term of a series, the $r-m+t$ being considered as one quantity. So likewise we may say, the mth order, the $(m-1)$th order, &c.

CHAPTER I.

SECTION I.

OF AN ASCENDING SYSTEM OF SERIES.

9. From Art. 4, it is evident that, in order to ascertain the the properties of any *system* of series, the investigation should proceed in accordance with the *law*, by which the successive *orders* arise from its basis.

10. The law of succession, relating to the system of series, which is the subject for discussion in this chapter, is supposed to be such, that *the sum of* n *terms of any order in the system, is equal to the* n*th term of the* next *succeeding order.*

Let the *number* of any order in the system, be denoted by m, then, when $m-1$ is not less than 1, the *law of succession* may be generally expressed thus:

THE SUM OF n TERMS OF THE $(m-1)$th ORDER OF SERIES, IS EQUAL TO THE nth TERM OF THE mth ORDER OF SERIES.

Corollary.—The first term of the mth order will be the same as that of the $(m-1)$th order; and the first term of each ascending order will be the same as that of the basis.

11. Let a, b, c, d, e, ... u, v, be n successive terms of *any order* of series in a system, arranged in alphabetical order from left to right, a being the first term. (Art. 6.)

12. Also, let A, B, C, D, E, ... U, V, be n successive terms of the *next succeeding order*, derived from the preceding order (Art. 11,) according to the LAW *of succession*, (Art. 10,) A being the first term.

13. The successive values of the several terms pertaining to the order in Art. 12, may be expressed by their respective equivalents (taken from the order in Art. 11, according to Art. 10,) as shown in the following diagram, thus:

First term	"	$A = a$,
2d	"	$B = a + b$,
3d	"	$C = a + b + c$,
4th	"	$D = a + b + c + d$,
"	"	" " " " " " " " "
"	"	" " " " " " " " "
$(n-1)$th	"	$U = a + b + c + d + \ldots\ldots u$,
nth	"	$V = a + b + c + d + \ldots\ldots u + v$.

Hence, each term of the series in Art. 12, is equivalent to a *rank of terms* taken from the series in Art. 11, and (numbering

the ranks in the diagram, downwards,) *any rank*, as the nth, *contains* (according to Art. 10,) n terms; and it is obvious, that *the sum of* n *of these ranks, is equal to the sum of* n *terms of the series in* Art. 12.

14. The series in Art. 11, is assumed to be *general*, or *any series* whatever, and therefore includes the series in Art. 12; hence, in the same manner that the *former series* produced the *latter*, the *latter* may produce a *third order*, and so on, up to the (m—1)th order, which may likewise produce the mth order of series.

SECTION II.

LEMMA I.

15. Let it be supposed that several ranks of the assumed series in Art. 11, each having an equal number of terms, are arranged (according to Art. 5,) parallel to, and at equal distances from, each other, in such a manner that the like terms, in the respective ranks, shall form *files* perpendicular to the ranks; as in Fig. 1, Art. 18. From the very nature of the construction of such a diagram, *it will, evidently, have as many files, as there are terms in each rank, and as many ranks, as there are terms in each file.*

16. In a diagram constructed according to the foregoing directions (Art. 15,) the *ranks* will be numbered from the top downwards, and the *files* from left to right.

Explanation of terms, which will be used in the course of this Lemma.

17. To construct a diagram Rank and File, is to construct one as directed in Art. 15.

An Ordinate Term in a diagram constructed *rank* and *file*, is that term which is common to a specified *rank* and *file*.

An Ordinate Rank does not include the *ordinate term*, but is that part of the *rank* to the left of it.

An Ordinate File includes the *ordinate term* and that part of the file of the diagram which is above that term.

The *ordinate rank* and *ordinate file*, which are reckoned from the same *ordinate term*, are said to *correspond to each other*.

Example.

If in Fig. 1, Art. 18, d be taken for the *ordinate term* of the 3d rank and 4th file; then, a, b, c, form the *ordinate rank*, and d, d, d, the *ordinate-file*.

18. Let a *diagram* be constructed *rank* and *file*, (Art. 17,) from the assumed series in Art. 11, to *any* extent, such that the number of *files* shall be *one* greater than the number of *ranks*. Let the number of ranks be n, and the number of files, $n+1$. Let Fig. 1 represent the supposed *diagram*.

It is now proposed to show some relations which exist between the several parts of a diagram constructed as the one here supposed, (Fig. 1.)

Fig. 1.

a	b	c	d	e
a	b	c	d	e
a	b	c	d	e
a	b	c	d	e

Let us suppose m to be equal any number of units, not less than *one*, nor greater than n; and also suppose the $(m+1)$th term of the mth rank, to be the *ordinate term* of this rank; (Art. 17,) then, by Art. 17, the mth *ordinate rank* contains m terms; for this *rank* consists of the terms to the left of the *ordinate term*, which is, by hyp., the $(m+1)$th term; and, also, the corresponding *ordinate file* contains m terms, for this file includes the *ordinate term*, (Art. 17,) which, by hyp., is in the mth rank, and it also includes all the terms in the $(m+1)$th file above the ordinate term; that is, this *ordinate file* contains one term in each of the m ranks, thus making m terms; hence,

19. ***The number of terms, in the* m*th ordinate rank, is the same as in its corresponding ordinate file; that is, each contains* m *terms.***

20. Now let m be taken successively for all the integer numbers from 1 to n; then it is obvious that as m is successively augmented by units, the $(m+1)$th term of the mth rank (which is the ordinate term by hyp.) will be successively in all the ranks of the diagram, from the top downwards (Art. 16,) and also successively, in all the files (except the first) from left to right. Hence, it is manifest,—1. That each *ordinate rank* has a corresponding *ordinate file* of an equal number of terms, and vice versa.—2. That no term in the first file of the diagram can be an *ordinate term*; for when $m=1$, then $m+1=2$, which is by hyp. the *ordinate term* for the first rank.—3. That the number of terms in any *ordinate rank*, is equal to the *number* of said *rank*; for, from the foregoing, the mth *ordinate rank*, and also, its corresponding *ordinate file*, which (by sup.) is part of the $(m+1)$th file, contains m terms.

21. Let a diagonal be drawn through the diagram (Fig. 1, Art. 18,) so as to separate each *ordinate rank* from its corresponding *ordinate file*, leaving the *former* below, or to the left of the diagonal, and the *latter* above, or to the right of it, as in Fig. 1. Then there being the same number of *ordinate ranks* as *ordinate*

files, (Art. 20,) and the same number of terms in each of the *former*, as in its corresponding *latter*; (Art. 19,) it is evident, that, in the *diagram*, the same number of terms will be on one side of the diagonal, that is on the other, and that the first file of the diagram will be wholly to the left, and the last file wholly to the right of the diagonal.

22. The nth *ordinate rank* under the diagonal (in Fig. 1,) has n terms, (Art. 20,) also, the nth rank in the diagram of Art. 13, has n terms; therefore, the successive ranks under the former condition, are respectively equal to those under the latter, and consequently the *sum* of n ranks of the former, is equal to the *sum* of n ranks of the latter; Hence,

If, of any order of series (as that in Art. 11,) *a diagram be constructed, and a diagonal be drawn, as directed in Arts.* 21 *and* 18, *the total amount of terms under the diagonal*, i. e. the sum of n *ordinate ranks*, *will be equal to the sum of* n *terms of the next succeeding order of series*, (as that in Art. 12.)

SECTION III.

LEMMA 2.

23. Let it be supposed that *two diagrams* are constructed, *rank* and *file*, (Art. 17,) to any extent whatever; provided the number of *files*, in *each*, exceed the number of ranks by one, (as in Art. 18); and let a *diagonal* pass through each *diagram* as directed in Art. 18. Let the *first diagram* be constructed of the supposed series in Art. 11, and the *second* of the supposed series in Art. 12. Moreover, let p be any number whatever. Then,

If the first diagram be such, that the sum total of all the terms in it, is p *times the sum total of all the terms below the diagonal*, it is to be proved that, *the sum total of all the terms in the second diagram, will be* $p+1$ *times the sum total of all the terms below its diagonal.*

Fig. 2.

a	b	c	d	e
a	b	c	d	e
a	b	c	d	e
a	b	c	d	e

Fig. 3.

A	B	C	D	E	"	"	V	W
A	B	C	D	E	"	"	V	W
A	B	C	D	E	"	"	V	W
A	B	C	D	E	"	"	V	W
"	"	"	"	"		"	"	"
"	"	"	"	"	"	"	"	"
A	B	C	D	E	"	"	V	W
A	B	C	D	E	"	"	V	W

According to the above supposition, let the *first diagram* be represented by Fig. 2, and the *second* by Fig. 3. Also, let m denote any number of units, not less than one. Furthermore, concerning Fig. 2, let us suppose that

$m =$ the number of ranks,
$m + 1 =$ the number of files,
$x =$ the sum total of all the terms in the diagram, and
$y =$ the sum total of all the terms below the diagonal.

And, concerning fig. 3, let

$X =$ the sum total of all the terms in the diagram,
$Y =$ the sum total of all the terms below the diagonal;
$X - Y =$ (evidently) the amount above the diagonal.

Then, if $x = p\,y$, which is supposed, it is to be proved that $X = (p + 1)Y$.

Demonstration.—By hyp. Figs. 2 and 3 are constructed like Fig. 1 in Lemma I; therefore, what is there proved is applicable in the present demonstration. Consequently;—1. The mth *ordinate rank*, in Fig. 3, contains m terms, (Art. 20,) the sum of which terms is equal to y, (Art. 22.)—2. The *ordinate file*, which corresponds to the mth *ordinate rank*, also contains m terms, (Arts. 19, 20,) each of which is the $(m+1)$th term in its respective *full rank* of the diagram; for *each* is in the $(m + 1)$th file; and therefore (by Arts. 10, 13,) it is equal to the sum of $m + 1$ terms of the next preceding order of series; that is, equal to one *full rank* in Fig. 2, which by sup. has $m + 1$ terms. But by hyp. there are m ranks in Fig. 2; hence, the said *ordinate file* in Fig. 3, which has m terms, is equal to x.

Now by sup. $x = p\,y$; hence we have proved that in Fig. 3, any *ordinate file*, as the mth is equal to p times its corresponding ordinate rank; (for m is general, Art. 8.) But there are as many ordinate files, as ordinate ranks, (Art. 20,) consequently it is evident that the sum of all the ordinate files in Fig. 3, which is equal to $X - Y$, is equal to p times the sum of all the ordinate ranks; that is, equal to pY. Hence, $X - Y = pY$, or $X = (p+1)Y$.

Q. E. D.

24. The foregoing demonstration is of the most general character; therefore, if what is assumed in Fig. 2, (viz. $x = py$,) which is constructed of the series in Art. 11, be true; the same is also *proved* to be true of Fig. 3, constructed of the series in Art. 12. Consequently, by virtue of the demonstration, the same *property* is proved to be true of every succeeding diagram, similar to Fig. 3, which shall be constructed from the succeed-

ing orders of series which shall arise by Art. 10, from the series in Art. 11: (See Art. 14.)

Again, by the *demonstration*, the number denoted by p, as applied to *any* order of series, is augmented by a unit, when taken in the next succeeding order. Now in accordance with the general supposition, let p answer to the *basis*, or first order in the system; (see Art. 3.) also, in conformity with the foregoing, pass, by successive steps, from the first to the $(m+1)$th order, and from the first to the $(m-1)$th order, and it will be evident that when applied to a diagram, (like Fig. 3,) constructed of the $(m-1)$th order of series, p will have been augmented by $m-2$ units; therefore the sum total of this diagram will be $p+m-2$ times the amount below the diagonal.

25. Let a diagram similar to Fig. 3, having n ranks, and $n+1$ files, be constructed of the $(m-1)$th order of series. Also, let $W=$ the sum total of all the terms in said diagram, and $S=$ the sum of n terms of the mth order of series.

Now, by Art. 22, the amount under the diagonal, in this diagram, is equal to S, and, by Art. 24, it is one $(p+m-2)$th part of W, hence $W=(p+m-2)S$, or

$$S=\frac{W}{P+m-2}$$

SECTION IV.

SUMMATION OF SERIES.

To find the sum of n *terms of the* m*th order of series.* [See Art. 10.]

Of the $(m-1)$th order of series, suppose a diagram (similar to Fig. 3 in Art. 23,) to be constructed, consisting of n ranks and $n+1$ files. Then, as each rank of the diagram consists of $n+1$ terms of the $(m-1)$th order, (Art. 15,) and, as there are n such ranks in the diagram by hyp.; it is evident that the whole amount of the diagram is equal to the *sum* of $n+1$ terms of the $(m-1)$th order, multiplied into n, the number of ranks. But this *whole amount* is equal to W, in Art. 25. Hence;

To find the SUM *of* n *terms of the* m*th order of series*, observe the following

26. RULE. *Multiply the sum of* $n+1$ *terms, of the* $(m-1)$*th order of series, into* n, *and divide the product by* $p+m-2$; *the quotient will be the required sum.*

27. The above *rule* is general, and comprehends *any order* of *any system* of series (Art. 3) arising according to Art. 10; provided the *basis*, or *first order* of the system, be such, that, when a *diagram* of an indefinite extent is constructed according to the directions in Art. 18, the *whole amount* of the diagram shall be a *certain number* of times greater than the part under the diagonal. But it remains for us to show that this *proviso* (Art. 23,) is not wholly imaginary. To this end,

28. Let us suppose, for the *basis* or *first order* of a system of series, *any series*, such that all the terms of it shall be equal to each other. And let a be one of those terms.

Of this series, suppose a diagram to be constructed according to Fig. 1, Art. 18, having n ranks and $n+1$ files. Then, the whole number of terms above the diagonal equals those below it, (Art. 21) and by sup. the terms are equal to each other; consequently the whole amount above the diagonal equals that below it, or the total amount of the diagram is 2 times the amount below the diagonal. Therefore, the investigation of Lemma 2, is applicable to a *system of series*, whose *basis* or *first order* of series consists of terms equal to each other; consequently, in this case, as has just been proved, p, as assumed in Art. 23, becomes 2. Therefore, in the *Rule* (Art. 26), the divisor, $p + m - 2$, becomes $2 + m - 2 = m$. Hence,

29. When the *basis* of a system of series consists of equal terms, according to the supposition in Art. 28, the general *rule* in Art. 26, becomes modified, thus:

To find the sum of n *terms of the* mth *order of series, when* m *is not less than* 2:

RULE. MULTIPLY THE SUM, OF $n+1$ TERMS, OF THE $(m-1)$th ORDER OF SERIES, INTO n; AND DIVIDE THE PRODUCT BY m.

Or thus:

MULTIPLY THE SUM OF $n+1$ TERMS OF THE NEXT PRECEDING ORDER, INTO n; AND DIVIDE THE PRODUCT BY THAT NUMBER WHICH DENOTES THE ORDER OF SERIES.

Scholium. It is evident that m, as taken above, should not be less than 2, for then, $m - 1$ would be less than 1, and we are supposed to commence the summation of the several orders of series, at the first order.

30. In the following algebraic formula, respecting the summation of *any order of series*, let S denote the sum of n terms, and S' the sum of $n+1$ terms of the next preceding order.

Now if we have the value of S, in terms, or functions of n (the number of terms,) for any order of series; it is evident that in order to obtain the sum of $n+1$ terms of the same order, we have only to substitute in the given value, or formula, $n+1$, for n, because this is the official nature of n by hyp. (Art. 8.)

Hence S, for any order, may be changed to S' for the next succeeding order, by substituting, in the value of S, $n+1$ for n.

By supposition in Art. 28, each term of the first order of series is equal to a; therefore the sum of n terms is n times a, or equal to an.

As m denotes the number of the order (by sup.) we may, at pleasure, for m, substitute its arithmetical value.

1st Order of Series.

31. By sup. each term equals a, and by Art. 30,

$$S = an, \text{ or } S = a \,.\, \frac{n}{1}$$

2d Order of Series.

32. By Arts. 29 and 30, $S = S' . n \div m$, but by Art. 30, $S' = a \,.\, \frac{n+1}{1}$ Hence, $S = a \,.\, \frac{n+1}{1} n \div m$, and by permutation of factors, and subst. for m, $S = a \,.\, \frac{n(n+1)}{1 \,.\, 2}$

3d Order of Series.

33. By Arts. 29 and 30, $S = S' . n \div m$, but (Art. 30.) $S' = a \,.\, \frac{(n+1)\,(n+2)}{1 \,.\, 2}$ Hence $S = a \,.\, \frac{(n+1)\,(n+2)}{1 \,.\, 2} n \div 3$, and by permutation of factors, $S = \frac{n\,(n+1)\,(n+2)}{1 \,.\, 2 \,.\, 3}$

4th Order of Series.

34. By Arts. 29 and 30, $S = S' . n \div m$, but

$$S' = a \,.\, \frac{(n+1)\,(n+2)\,(n+3)}{1 \,.\, 2 \,.\, 3}, \text{ and } m = 4.$$

Hence, $S = a \,.\, \frac{n\,(n+1)\,(n+2)\,(n+3)}{1 \,.\, 2 \,.\, 3 \,.\, 4}$

5th Order of Series.

35. By Arts. 29 and 30,

$$S=S'.n\div m=\text{(Art. 30) } a.\frac{(n+1)(n+2)(n+3(n+4)}{1\ .\ 2\ .\ 3\ .\ 4}.n\div 5.$$

$$\text{Hence, } S=a.\frac{n(n+1)(n+2((n+3)(n+4)}{1\ .\ 2\ .\ 3\ .\ 4\ .\ 5}.$$

36. By inspecting the formulæ, in the five preceding articles, we see that the respective values of S, pertaining to the first five orders of series (Art. 3,) possess *certain general properties;* as—1. The *number* of factors which contain n, in the *numerator*, and also the *number* of factors in the *denominator*, is equal to the *number* which denotes the respective order of series. 2. Each factor containing n, in the numerator, is a unit less than the factor next to its right, and the least factor is on the extreme left, and is equal to n, while the largest factor is on the extreme right, and is equal to n augmented by a number of units one less than the number of the order to which the formula is applied. 3. Each factor in the denominator, is a unit less than that next to its right, that on the extreme left being only a unit, while that on the extreme right is equal to the number of the order pertaining to S. 4. The first term (a) of the 1st order of series is an independent factor or co-efficient, to the fraction in functions of n.

Now it is to be proved that the four preceding *general properties* are possessed by the *formula* expressing the value of S in the mth order of series; to do which,

Let us suppose that the *properties*, which we have seen to be related to the first five orders of series, are also related to all the orders up to the bth order; then these *properties* also relate to the next higher order; that is, to the $(b+1)$th order. For according to the sup. respecting the

bth Order of Series,

We have

$$S=a.\frac{n(n+1)(n+2)(n+4)\ .\ .\ .\ (n+b-1)}{1\ .\ 2\ .\ 3\ .\ 4\ .\ .\ .\ .\ b} \quad (1)$$

Now in order to obtain the value of S in the

$(b+1)$th Order of Series,

We have by Arts. 29 and 30,

$S=S'.n\div(b+1)$; but by Art. 30, we have

$$S' = a\,\frac{(n+1)\,(n+2)\,(n+3)\,(n+4)\;.\;.\;.\;(n+b)}{1\;.\;2\;.\;3\;.\;4\;.\;.\;.\;.\;b} \quad (2)$$

By substituting for S', its value, we have

$$S = a\,.\,\frac{n\,(n+1)\,(n+2)\,(n+3)\,(n+4)\;.\;(n+b)}{1\;.\;2\;.\;3\;.\;4\;.\;.\;.\;b\,.\,(b+1)}$$

or more general, thus:

$$S = a\cdot\frac{n\,(n+1)\,(n+2)\,(n+3)\,(n+4)\;.\;.\;.\;(n+b)}{1\;.\;2\;.\;3\;.\;4\;.\;5\;.\;.\;.\;.\;(b+1)}$$

Now in passing from the bth to the $(b+1)$th order, each factor in the numerator of S' has been augmented by 1, (Art. 30) and the *numerator* and *denominator* have each received a new factor (Art. 29;) hence, from the foregoing, it is evident that these *four general properties* also pertain to S, in the $(b+1)$th order.

Furthermore, b may be taken successively for all the integer values from one up to $m-1$ units (Art. 8;) hence the *four properties* related to the first five orders, are proved to relate to all the orders up to the $(m-1)$th order, and by virtue of the demonstration, up to the mth order. Q. E. D.

37. *Hence, for the* m*th order of series, we have the*

GENERAL FORMULA.

$$S = a\,.\,\frac{n\,(n+1)\,(n+2)\,(n+3)\,(n+4)\;.\;(n+m-1)}{1\,.\,2\;.\;3\;.\;4\;.\;5\;.\;.\;.\;.\;m}$$

38. *To find the* n*th term of the* m*th order of Series,*

Let T denote the nth term; then (by Art. 10,) T equals the sum of n terms of the $(m-1)$th order, which sum may be expressed by the general formula in Art. 37, by substituting $(m-1)$ for m. Hence, for the value of T, we have the

GENERAL FORMULA.

$$T = a\,.\,\frac{n\,(n+1)\,(n+2)\,(n+3)\,(n+4)\;.\;(n+m-2)}{1\,.\,2\;.\;3\;.\;4\;.\;5\;.\;.\;.\;.\;(m-1)}$$

Scholium. The value of T for the *first order* of series, cannot be obtained by the *formula;* for in this case $m-1$, in the denominator becomes $1-1=0$; but by sup. (Art. 28,) each term in the 1st order equals a; hence, by sup. $T=a$.

39. *Let* m, n, *and* z, *be any positive integer-numbers whatever, then*

It is required to find the sum of a series of terms, in an Ascending Order, such, that the number of terms shall be z — 1 *units less than* n;

and that the number of the Order shall be z — 1 *units greater than* m, *i. e., so that the number of terms shall be as many units less than* n, *as the number of the Order is units greater than* m.

Now by sup., the number of terms $= n - (z - 1)$, and the No. of the ascending order $= m + z - 1$; therefore to find the required *sum*, we have only to substitute in the *general formula*, (Art. 37,) $n - (z - 1)$ for n, and $m + z - 1$ for m. But by substitution, $n + m - 1$, in Art. 37, becomes $[n - (z-1)] + [m + (z - 1)] - 1 = n + m - 1$, the factor on the extreme right of the numerator, while that on the left will be by sup.

Now in the expression for the *sum*, if the order of the factors in the numerator be reversed, the factor on the extreme left will be $n + m - 1$, and that on the right $n - (z - 1)$; moreover, each *factor* will be a unit greater than that next to its right, which is evident by Art. 36, (2). Hence, the

GENERAL FORMULA,

$$S = a\frac{(n+m-1)\,(n+m-2)\,(n+m-3)\ \ .\ \ .\ \ [n-(z-1)]}{1\ \ .\ \ 2\ \ .\ \ 3\ \ .\ \ .\ \ .\ \ (m+z-1)} \qquad (1)$$

39½. Cor. *When the number of terms is as many units less than* n, *as the number of the ascending order is units higher than the first order, or* 1; then, $m = 1$ by supposition, and consequently the first factor of the numerator becomes $n + 1 - 1 = n$, and the last factor of the denominator becomes $1 + z - 1 = z$. Hence, under the present case we have for

A GENERAL FORMULA

$$S = a\,\frac{n(n-1)\,(n-2)\,(n-3)\,(n-4)\ \ .\ \ [n-(z-1)]}{1\ \ 2\ \ .\ \ 3\ \ .\ \ 4\ \ .\ \ 5\ \ .\ \ .\ \ z} \qquad (2)$$

Practical Examples.

The following is an Ascending System of Series, consisting of eight orders, arising (by Art. 10,) from a basis, each term of which equals a unit; thus—

1st Order,	1	1	1	1	1	1	1	1
2d "	1	2	3	4	5	6	7	8
3d "	1	3	6	10	15	21	28	36
4th "	1	4	10	20	35	56	84	120
5th "	1	5	15	35	70	126	210	330
6th "	1	6	21	56	126	252	462	792
7th "	1	7	28	84	210	462	924	1716
8th "	1	8	36	120	330	792	1716	3432

Let n, m, S, and T, be the No.^{s} of terms, No. of the order, sum of n terms, and the nth terms respectively, as heretofore designated; then, by the general formulæ in Arts. 37 and 38, we obtain the following results;

Example 1. When $n = 13$, and $m = 1$, to find S and T,

$$S = 1 \cdot \frac{13}{1} = 13, \text{ and } T = 1. \quad \text{(Schol. Art. 38.)}$$

Ex. 2. When $n = 13$, and $m = 2$, we have

$$S = 1 \cdot \frac{13\,(13+1)}{1 \cdot 2} = 91, \text{ and } T = 1 \cdot \frac{13}{1} = 13.$$

Ex. 3. When $n = 13$, and $m = 3$, we have

$$S = 1 \cdot \frac{13\,(13+1)\,(13+2)}{1 \cdot 2 \cdot 3} = 455, \text{ and } T = 1 \cdot \frac{13(13+1)}{1 \cdot 2} = 91.$$

Ex. 4. When $n = 17$, and $m = 4$, we have

$$S = 1 \cdot \frac{17\,(17+1)\,(17+2)\,(17+3)}{1 \cdot 2 \cdot 3 \cdot 4} = 4845, \text{ and}$$

$$T = 1 \cdot \frac{17\,(17+1)\,(17+2)}{1 \cdot 2 \cdot 3} = 969.$$

Ex. 5. When $n = 20$, and $m = 5$, $S = 42504$, and
When $n = 8$, and $m = 5$, we have $T = 330$.

NOTE. The foregoing examples are, no doubt, sufficient to illustrate, and show the application of, the general formulæ in Arts. 37 and 38.

CHAPTER II.

OF A DESCENDING SYSTEM OF SERIES. (*See Art.* 3.)

SECTION I.

40. Any *assumed series*, taken for the basis of a *Descending System*, is termed the PRIMITIVE ORDER *of Series*, (or *Prim. Order.*)

41. The *derivative orders*, proceeding from the basis, by the *Law* soon to be established, are termed *Orders of Differences*, and are numbered according to their respective distances from the *Prim. Orders;* as 1st Diff., 2d Diff., 3d Diff., nth Diff., &c.

42. The several *orders of differences* are to be derived as follows, viz:—The 1st order of Diff. from the *Prim.* Order; the 2d order from the 1st order; the 3d order from the 2d order; and so on up to the mth order, which is derived from the $(m-1)$th order of Diff.

43. In reference to the *whole system*, the *Prim.* Order may be termed the 1st Order of the System, and the mth Diff. will then be the $(m+1)$th Order of the System.

GENERAL LAW OF SUCCESSION,

By which any Order in the System is derived.

44. IN ANY ORDER OF THE SYSTEM, SUBTRACT THE nth TERM FROM THE $(n+1)$th TERM, AND THE REMAINDER, WILL BE THE nth TERM OF THE NEXT SUCCEEDING ORDER.

Example.

Prim. Order, or 1st Order,	1,	8,	27,	64,	125,	216,
1st Diff. " 2d Order,	7,	19,	37,	61,	91,	
2d Diff. " 3d Order,	12,	18,	24,	30,		
3d Diff. " 4th Order,	6,	6,	6,			
	0,	0.				

45. Let A, B, C, E, F, *&c.*, be the successive terms of any series whatever, in alphabetical order from left to right, A being the first term. And let this series be taken for the basis of a *General Descending System of Series*, as represented by the following *Diagram*, Art. 46.

Moreover, let the successive *orders of Differences* be characterised by the successive letters in the alphabet and in the same order; that is, let the 1st Diff. be characterised by a, the 2d Diff. by b, and so on.

Furthermore, in each *Order of Differences*, let the arrangement of the successive terms, be denoted by the *small letter* attached to the right of each term. Thus, take any order of Diff. as the 4th, then $\mathbf{d}_a$, $\mathbf{d}_b$, $\mathbf{d}_c$, . . . $\mathbf{d}_n$, denote the 1st, 2d, 3d and nth terms of this order of Differences.

Lastly, Let the successive orders of the *system* be arranged in ranks under each other, according to the order of their respective derivations, and in such a manner, that each term of any Diff. shall be *below*, and *midway between* the two terms of the preceding order of which it is the *Difference*.

46. The following Diagram represents a *Descending* System of Series, constructed according to the foregoing Arts. 44, 45. p may be taken, as any integer from 1, up to z units. Then we have the

Diagram.

Prim. Ord.	A,	B,	C,	D,	E,	F,	G,	H,	J,	K,	&c.
1st Diff.	$\mathbf{a}_a$,	$\mathbf{a}_b$,	$\mathbf{a}_c$,	$\mathbf{a}_d$,	$\mathbf{a}_e$,	$\mathbf{a}_f$,	$\mathbf{a}_g$,	$\mathbf{a}_h$,	$\mathbf{a}_i$,		&c.
2d Diff.	$\mathbf{b}_a$,	$\mathbf{b}_b$,	$\mathbf{b}_c$,	$\mathbf{b}_d$,	$\mathbf{b}_e$,	$\mathbf{b}_f$,	$\mathbf{b}_g$,	$\mathbf{b}_h$,	$\mathbf{b}_i$,		&c.
3d Diff.	$\mathbf{c}_a$,	$\mathbf{c}_b$,	$\mathbf{c}_c$,	$\mathbf{c}_d$,	$\mathbf{c}_e$,	$\mathbf{c}_f$,	$\mathbf{c}_g$,	$\mathbf{c}_h$,			&c.
4th Diff.	$\mathbf{d}_a$,	$\mathbf{d}_b$,	$\mathbf{d}_c$,	$\mathbf{d}_d$,	$\mathbf{d}_e$,	$\mathbf{d}_f$,	$\mathbf{d}_g$,	$\mathbf{d}_h$,			&c.
" "	"	"	"	"	"	"	"				"
" "	"	"	"	"	"	"	"				"
pth Diff.	$\mathbf{k}_a$,	$\mathbf{k}_b$,	$\mathbf{k}_c$,	$\mathbf{k}_d$,	$\mathbf{k}_e$,	$\mathbf{k}_f$,					&c.
$(p+1)$th Diff.	$\mathbf{l}_a$,	$\mathbf{l}_b$,	$\mathbf{l}_c$,	$\mathbf{l}_d$,	$\mathbf{l}_e$,	$\mathbf{l}_f$,					&c.
" "	"	"	"	"	"	"					"
" "	"	"	"	"	"	"					"
$z - 1$th Diff.	$\mathbf{w}_a$,	$\mathbf{w}_b$,	$\mathbf{w}_c$,	$\mathbf{w}_d$,							&c.

47. It is now proposed to show the relation that exists between the successive orders, in the foregoing diagram, (Art. 46,) to do which,

Let any two consecutive orders, as the pth, and the $(p+1)$th orders of Diffs. be taken; then by Art. 44, the values of the $(n-1)$ successive terms in the $(p+1)$th order of Diff. may be expressed by the $(n-1)$ successive equations, arranged *rank* and *file*, as in the subjoined diagram, thus:

Equation	1st	$\mathbf{k}_b - \mathbf{k}_a = \mathbf{l}_a$
"	2d	$\mathbf{k}_c - \mathbf{k}_b = \mathbf{l}_b$
"	3d	$\mathbf{k}_d - \mathbf{k}_c = \mathbf{l}_c$
"	3d	$\mathbf{k}_e - \mathbf{k}_d = \mathbf{l}_d$
"	4th	$\mathbf{k}_f - \mathbf{k}_e = \mathbf{l}_e$
"	"	"
"	"	"
"	$(n-1)$th	$\mathbf{k}_n - \mathbf{k}_{n-1} = \mathbf{l}_{n-1}$

Now in producing the $n-1$ terms of the $(p+1)$th order, as is exhibited in the above diagram, all the terms of the pth order, up to the nth, are taken positively, except the first; and all likewise, except the nth term, are taken negatively; hence, by adding the successive equations together, the like *positive* and *negative* quantities will annihilate each other, and we have

$$\mathbf{k}_n - \mathbf{k}_a = (\mathbf{l}_a + \mathbf{l}_b + \mathbf{l}_c + \mathbf{l}_d + \mathbf{l}_e \ . \ \ . \ \ . + \mathbf{l}_{n-1}) \quad . \quad (1)$$

That is, *In the* p*th order, the* n*th term minus the first, equals the sum of* $(n-1)$ *terms of the* $(p+1)$*th* order of Diff.

Or by transposition, we have

$$\mathbf{k}_n = \mathbf{k}_a + (\mathbf{l}_a + \mathbf{l}_b + \mathbf{l}_c + \mathbf{l}_d + \mathbf{l}_e + \ \ . \ \ + \mathbf{l}_{n-1}) \quad . \quad (2)$$

That is, *The* n*th term of the* p*th order equals the first term, plus the sum of* $(n-1)$ *terms of the* $(p+1)$*th* order of Diff.

48. Now for n, we may substitute all integer values from 1, to n (Art. 8,); therefore, by Art. 47, (1), we obtain the n successive equations, which are arranged, *rank and file*, as in the following Diagram.

Again; by the construction of the following diagram of equations,—1. The *first members* of the n equations, constitute *two files*, one of which consists of the n successive terms of the pth order, *all positive;* and the other file consists of n terms, each of which is the first term of the pth order, and is *negative.* —2. The *second members* of the equations in the diagram, are (evidently from Arts. 10 and 13,) the successive values of the $(n-1)$ successive terms, in the *next succeeding Ascending order of series*, arising (by Art. 10) on the $(p+1)$th order as a *basis.* [This *Ascending order* can contain but $(n-1)$ terms, for the *first* equation has zero for its 2d member.]

Hence, the Diagram alluded to above, will stand as follows:

Equation	1	$\mathbf{k}_a - \mathbf{k}_a = 0,$
"	2	$\mathbf{k}_b - \mathbf{k}_a = \mathbf{l}_a,$
"	3	$\mathbf{k}_c - \mathbf{k}_a = \mathbf{l}_a + \mathbf{l}_b,$
"	4	$\mathbf{k}_d - \mathbf{k}_a = \mathbf{l}_a + \mathbf{l}_b + \mathbf{l}_c,$
"	5	$\mathbf{k}_e - \mathbf{k}_a = \mathbf{l}_a + \mathbf{l}_b + \mathbf{l}_c + \mathbf{l}_d,$
"	"	" " " " " " "
"	"	" " " " " " "
"	n	$\mathbf{k}_n - \mathbf{k}_a = \mathbf{l}_a + \mathbf{l}_b + \mathbf{l}_c + \mathbf{l}_d, + \ . \ + \mathbf{l}_{n-1}.$

Let S represent the sum of n successive terms in the pth order, which may be considered the *basis*, or *first order* of an *Ascending System of Series*, arising under Art. 10; also, let M

denote the sum of n successive terms of a *series*, every term of which is equal to the first term of the pth Diff., and let said series be considered the *basis*, or *first order*, of any *Ascending System of Series* (Art. 10); moreover, let the $(p+1)$th Diff., be considered the *basis*, or *first order*, of an *Ascending System of Series*, (Art. 10,) and let Q denote the sum of $(n-1)$ terms of the second order of this ascending system: Then,

By adding together (equals to equals) the several equations in the above diagram, and *transposing*, we obtain, when S, M and Q, are substituted, the following equation,

$$S = M + Q.$$

49. Let the six following ranks of letters represent six ranks of Series, such that the successive terms of each rank shall be numbered from left to right, according to the order of the small adjacent letters, as in Art. 45.

A_a,	A_b,	A_c,	A_d,	A_e,	A_f,	A_g,	A_h,	Rank (I)
B_a,	B_b,	B_c,	B_d,	B_e,	B_f,	B_g,	B_h,	" (II)
	C_a,	C_b,	C_c,	C_d,	C_e,	C_f,	C_g,	" (III)
$\mathfrak{a}_a$	$\mathfrak{a}_b$	$\mathfrak{a}_c$	$\mathfrak{a}_d$	$\mathfrak{a}_e$	$\mathfrak{a}_f$	$\mathfrak{a}_g$	$\mathfrak{a}_h$	" (IV)
$\mathfrak{b}_a$	$\mathfrak{b}_b$	$\mathfrak{b}_c$	$\mathfrak{b}_d$	$\mathfrak{b}_e$	$\mathfrak{b}_f$	$\mathfrak{b}_g$	$\mathfrak{b}_h$	" (V)
	$\mathfrak{c}_a$	$\mathfrak{c}_b$	$\mathfrak{c}_c$	$\mathfrak{c}_d$	$\mathfrak{c}_e$	$\mathfrak{c}_f$	$\mathfrak{c}_g$	" (VI)

Again, let us suppose that THREE Ascending Systems of Series, whose successive orders are derived by Art. 10, are formed the *first system*, from the pth Diff. (Art. 46) as a basis, the *second system*, from a basis every term of which is equal to the *first term* of the pth order of Diff., and the *third system*, from the $(p+1)$th order of Diff. (Art. 46) as a basis; and let the basis of each system be reckoned its *first order*.

Moreover,—1. In the *first system*, let rank (I) denote the rth order, and S, the sum of n, terms; also, let rank (IV) denote the $(r+1)$th order, and S', the sum of n terms.—2. In the *second system*, let rank (II) denote the rth order, and M, the sum of n terms; also, let rank (V) denote the $(r+1)$th order, and M', the sum of n terms.—3. In the *third system*, let rank (III) denote the $(r+1)$th order, and Q, the sum of $(n-1)$ terms; also, let rank (VI) denote the $(r+2)$th order, and Q' the sum of $(n-1)$ terms.

Now in accordance with the foregoing supposition respecting the terms S, S', M, M', Q, and Q',—if

$S = M + Q$; then, it is to be proved, that $S' = M' + Q'$.

Demonstration.—By sup. rank (IV) is the next ascending order above rank (I); therefore (Art. 10,) the sum of n terms of rank (I) equals the nth term of rank (IV), that is, $S = \mathbf{a}_n$. For the same reason, since rank (V) is the order next above rank (II), the sum of n terms (Art. 10,) equals $\mathbf{b}_n$, or $M = \mathbf{b}_n$. In like manner, since rank (VI) is by sup. one order above rank (III) (Art. 10,) the sum of $(n - 1)$ terms of rank (III) equals $\mathbf{c}_{n-1}$, or $Q = \mathbf{c}_{n-1}$; that is, $S = \mathbf{a}_n$, $M = \mathbf{b}_n$, and $Q = \mathbf{c}_{n-1}$; but by sup. $S = M + Q$, hence, by substitution, $\mathbf{a}_n = \mathbf{b}_n + \mathbf{c}_{n-1}$, . . . (3).

Or, *The* n*th term of rank (IV) equals the* n*th term of rank (V) plus the* (n — 1)*th term of rank (VI).*

Now let n be taken successively for all the integer numbers from 1 to n, and in the same successive order, let the above equation (3) be placed as in the following Diagram, i. e. so that the like terms shall successively fall under each other; then the first members of the several equations collectively, will constitute the n successive terms of rank (IV); and all the second members will constitute, 1st, a *file*, consisting of the n successive terms of rank (V,) and 2dly, a *file* consisting of the $(n - 1)$ successive terms of rank (VI). Therefore, taking S', M' and Q', according to supposition, we have

When	$n = 1$	$\mathbf{a}_a = \mathbf{b}_a + 0$ (for when $n = 1, Cn - 1 = 0$)
"	$n = 2$	$\mathbf{a}_b = \mathbf{b}_b + \mathbf{c}_a$
"	$n = 3$	$\mathbf{a}_c = \mathbf{b}_c + \mathbf{c}_b$
"	$n = 4$	$\mathbf{a}_d = \mathbf{b}_d + \mathbf{c}_c$
"	$n = 5$	$\mathbf{a}_e = \mathbf{b}_e + \mathbf{c}_d$
"	$n = 6$	$\mathbf{a}_f = \mathbf{b}_f + \mathbf{c}_e$
"	" "	" " "
"	" "	" " "
"	$n = n$	$\mathbf{a}_n = \mathbf{b}_n + \mathbf{c}_{n-1}$

Hence, by addition $S' = M' + Q'$ Q. E. D.

50. According to the above demonstration (Art. 49), if the equation, $S = M + Q$, pertaining to the rth ascending order of the pth order of Diff., *can be established*; then the equation, $S' = M' + Q'$, pertaining to the $(r + 1)$th ascending order of the pth order of Diff., *is also established.* But, according to Art. 48, the *former* equation *is* established, when $r = 1$; and therefore, the latter equation is established, when $r + 1 = 2$; and, as it is established when $r = 2$, it is also, when $r + 1 = 3$, and, in like manner, r may be taken, successively, for all integer

values (Art. 8,) from 1 to m; consequently, by virtue of the demonstration, we have, for the mth Ascending order, (arising by Art. 10, on the pth order of Diff, as a *basis*,) established the general formula,

$$S = M + Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

51. In the Descending System, (Art. 46,) it is evident by Art. 44, that the same relations exist between the Prim. Order, and the 1st order of Diff., that do between the pth and $(p+1)$th order of Diff.; therefore, when $p = 1$, it (p) may denote the number of the *Prim. Order*, instead of the 1st order of Diff., and according to this view, the $(z - 1)$th order of Diff. becomes the zth descending order of the system; for (Art. 8,) p may be taken, successively, for all the integer values from 1, to z units; *hence,*

1. *The successive orders in the Descending System may be numbered from the* 1st, or prim. order, *to the* z*th Descending Order.* And

2. *The Demonstration in* Art. 49, is *applicable to any two consecutive orders*, or, *successively, to every two consecutive descending orders* in the System. That is, equation (1) in Art. 50, *may be applied, successively, to all the orders in the Descending System, from the* 1st *to the* z*th Descending orders.*

52. From Art. 51, it is evident, concerning the general supposition in Art. 49, that what is *supposed* and *proved* relative to the pth and $(p+1)$th Diffs., is also *proved* of every two consecutive descending orders in said Descending System, (Art. 46) from the *first* (or prim. order) to the zth *descending order*. For, in integer values, p may be taken from 1 to z; and r, from 1 to m, (Art. 8); hence,

1. To every *descending order*, from the 1st (or prim. order) to the zth, may be ascribed an S,* and an M, which have the same functions, as those (S and M) were supposed to have, that belong to the pth Diff. [See Art. 49.]

2. The Q,* and the M, belonging to any given *descending order*, and pertaining to their respective *ascending systems of series*, [See Art. 49,] are the *sums* of the same number of terms, in their respective *ascending orders*, and the *numbers* of their *ascending orders*, are respectively the same.

3. Of any two consecutive descending orders, the Q, belonging to the *first*, is of an *Ascending order* a *unit lower*, and is the sum of a number of terms a *unit greater*, than is the Q, which

* The S and the Q are in effect synonymous terms, when applied to the same ascending order, arising by Art. 10, from any given order of Diff., (Art. 46.)

belongs to the *second*, or next succeeding descending order. [See Art. 49.] The same may be said of *M*, belonging to any two consecutive orders;—consequently,

The *M*, belonging to the *z*th descending order, is of an *ascending order*, $(z-1)$ units *higher*, and is the sum of a number of terms $(z-1)$ units *less*, than is the *M*, belonging to the Primitive order;—the same may be said of *Q*, in the *z*th dec. order.

53, 54, 55. Every *S*, except that belonging to the prim. order, will be replaced by *Q*. Furthermore, the *Q*, and the *M*, belonging to any *descending order*, will be denoted by the number of its respective *order*, placed adjacent to the right; thus, Q_1 (or *S*) and M_1 belong to the 1st (or prim.) order, Q_2 and M_2 belong to the 2d order; in like manner, Q_z and M_z belong to the *z*th, and Q_{z+1} to the $(z+1)$th *descending order:* therefore,

From what is proved, in Arts. 51 and 52, it is evident that the equation (1) in Art. 50, may, in a more general manner, be expressed thus:

$$Q_z = M_z + Q_{z+1} \qquad (1)$$

or, $$Q_z - Q_{z+1} = M_z \qquad (2)$$

56. Equation (2) in Art. 53, may, according to Art. 51, (2) be *successively applied* to all the descending orders belonging to the descending system in Art. 46, and, conformably to these *applications*, *z* takes successively all the integer values from 1 to *z* units, (Art. 8.)

Now, let the successive equations, corresponding to the successive orders, in the supposed system (Art. 46,) be arranged, *rank* and *file*, as exhibited in the following diagram.

For Q_1, which pertains to the prim. order, let *S* be substituted; then, in reference to the successive equations in the diagram, we have

Prim. Order,	or	1st	Order,	$S - Q_2 = M_1$
1st Diff.	"	2d	"	$Q_2 - Q_3 = M_2$
2d "	"	3d	"	$Q_3 - Q_4 = M_3$
3d "	"	4th	"	$Q_4 - Q_5 = M_4$
4th "	"	5th	"	$Q_5 - Q_6 = M_5$
5th "	"	6th	"	$Q_6 - Q_7 = M_6$
" "	"	"	"	" " "
" "	"	"	"	" " "
$(z-1)$th	"	*z*th	"	$Q_z - Q_{z+1} = M_z$

Now, in the above diagram, it is obvious from the nature of its supposed construction, that all the *Q*s from Q_1 (or *S*) to Q_z,

are taken *positively*, and that all from Q_2 to Q_{z+1}, are taken *negatively*.

Hence, let the equations, in the diagram, be added together, member to member, (*equals to equals*,) and, as the equal Qs, which are $+$, and $-$, destroy each other, we shall have, after transposing Q_{z+1}, the following

GENERAL EQUATION

for any Descending System whatever, to wit:

$$S = M_1 + M_2 + M_3 + M_4 + M_5 + \ldots + M_z + Q_{z+1} \qquad (3)$$

57. Suppose, that Q_1 and M_1 are respectively the sums of n terms of the mth ascending order; then by Art. 52, (3) it is evident, that M_z is the sum of $n - (z - 1)$ terms, and of an ascending order denoted by $m + (z - 1)$; that is, each of the successive Ms in equation (3), Art. 56, is of an ascending order, as many units *higher* than that of the first M, as its number of terms is *less*. Hence,

The value of each M may be expressed by the formula in Art. 39, in the following manner:

1. At each application of the formula, to the successive Ms, substitute for n in the formula, $n - (z - 1)$, which is a general expression for the number of terms in any M, as the zth M: Also, for m in the formula, substitute $m + (z - 1)$, which is a general expression for the number of any ascending order: hence $n + m$, in that formula, will constantly be, by substitution, $m + n$, for $n - (z - 1) + m + (z - 1) = m + n$.

2. To express any M, substitute, in the formula (Art. 39,) for the factor a, the first term of that descending order, to which said M, belongs;—the term on which M, may be said to arise. That is, for M_1, A_a takes the place of (a) in the formula; and for M_2, a_a; for M_3, b_a, and in like manner for M_z, w_a. [See the Diagram in Art. 46.] We may, however, drop the small letters adjacent to the right of the large ones.

3. In the successive applications, z will denote successively all integer values from 1 to z. Hence,

In accordance with the foregoing, equation 3, in Art. 56, may be expressed by the following

GENERAL FORMULA,

Which expresses the sum of n *terms of the* m*th ascending order, arising by Art.* 10, *on any Series whatever:*

$$S=\begin{cases} A\cdot\dfrac{(n+m-1)(n+m-2)(n+m-3)\,\ldots\ldots\,(n)}{1\times2\times3\times\ldots\ldots\times m} \\ +\mathbf{a}\cdot\dfrac{(n+m-1)(n+m-2)(n+m-3)\ldots\ldots(n-1)}{1\times2\times3\times\ldots\times(m+1)} \\ +\mathbf{b}\cdot\dfrac{(n+m-1)(n+m-2)(n+m-3)\ldots(n-2)}{1\times2\times3\times\ldots\times(m+2)}+\ldots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \ldots+\mathbf{w}\cdot\dfrac{(n+m-1)(n+m-2)\,\ldots\ldots\,[n-(z-1)]}{1\times2\times\ldots\times[m+(z-1)]} \\ +Q_{z+1} \end{cases}$$

58. By Art. 10, the nth term (which we will denote by T.) of the mth ascending order, is equal to the sum of n terms, of the $(m-1)$th ascending order; Hence, T may be expressed, by the general formula (Art. 57), if for m in the formula, we substitute $(m-1)$, because the formula will then be applied to the next lower order, which has the same number of terms, viz: $n-(z-1)$.

Now by applying the said formula, according to these directions, we obtain

A GENERAL FORMULA,

Which expresses the n*th term, of the* m*th ascending order, arising by Art.* 10, *on any Series whatever:*

$$T=\begin{cases} A\cdot\dfrac{(n+m-2)(n+m-3)(n+m-4)\,\ldots\ldots\,n}{1\times2\times3\times\ldots\times(m-1)} \\ +\mathbf{a}\cdot\dfrac{(n+m-2)(n+m-3)(n+m-4)\ldots\ldots(n-1)}{1\times2\times3\times\ldots\ldots\times m} \\ +\mathbf{b}\cdot\dfrac{(n+m-2)(n+m-3)(n+m-4)\ldots(n-2)}{1\times2\times3\ldots\ldots(m+1)}+\ldots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \ldots+\mathbf{w}\cdot\dfrac{(n+m-2)(n+m-3)\,\ldots\ldots\,[n-(z-1)]}{1\times2\times\ldots\ldots[m+(z-2)]} \\ +Q_{z+1} \end{cases}$$

General Remarks.

59. 1. Though the *Formulæ*, in Arts. 57 and 58, are of the most general character; yet, in *any application* of them, it is supposed that the value of Q_{z+1} *can be obtained,* which can be done, when the last, or zth Descending order consists of *equal terms*, for *then* the next succeeding order will consist of zeros, and of course $Q_{z+1}=0$.

2. In applying the said Formulæ, it is evident by Art. 57, (2) that the first term only, of each descending order, is necessary to be obtained. Now to procure the respective *first terms*, a sufficient number of terms should be taken from the first part of the prim. order, so that by taking the successive differences, *one term*, *at least*, of the zth, or last order, may be obtained. In order to insure ourselves more particularly respecting the nature of Q_{z+1}, (see Remark 1,) we should obtain some *two* or *three* terms in the last order of diff.

3. *Complete values* for S and T (Arts. 57 and 58,) may be obtained, when $Q_{z+1} = 0$, according to Remark 1.

4. *Approximate values* for S and T may be obtained, when Q_{z+1} is not equal to zero, provided, however, it be so small in decimal value, as to be dropped, or reckoned as zero, without affecting the values of S or T, within required decimal limits.

5. The value of any M, in equation (3) Art. 56, is either positive or negative, according to the sign of the *term* on which it arose (Art. 52.) This is evident from the nature of Art. 10, and consequently the general terms in the formulæ (Arts. 57 and 58,) will be either $+$, or $-$, according to the sign of the term substituted for (a) in the formula in Art. 39. [See Art. 57.]

***To obtain the sum of* n *terms of the Primitive Order of Series*, (See Arts. 40 and 46.)**

Proceed in determining the value of S, as directed in Art. 57, except, that in finding expressions for the successive Ms (Art. 56, (3)) apply formula (2) in Art. 39½, instead of formula (1) in Art. 39.

In this case M_z is of the zth ascending order, and is the sum of $n - (z - 1)$ terms.

Proceeding according to the directions just given, we obtain the following

GENERAL FORMULA.

$$S = nA + \mathbf{a} \cdot \frac{n(n-1)}{1 \times 2} + \mathbf{b} \cdot \frac{n(n-1)(n-2)}{1 \times 2 \times 3}$$

$$+ \mathbf{c} \cdot \frac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4} + \dots\dots\dots\dots\dots$$

$$\dots + \mathbf{w} \cdot \frac{n(n-1)(n-2)(n-3)(n-4) \dots [n-(z-1)]}{1 \times 2 \times 3 \times 4 \times 5 \times \dots \times z} + Q_{z+1}$$

Scholium 1. The foregoing is but a *particular formula*, embraced in the more general one, in Art. 57; consequently the *remarks* in Art. 59, are equally applicable to this formula.

Schol. 2. Though all the terms in the preceding formula, are denoted as positive, yet they *may* be negative, according to Art. 59, (5).

61. *To obtain an expression of the* nth *term of any Series*, for instance, *as the Primitive Order*, (Art. 46.)

It is evident, from equation (2), Art. 47, that the nth *term* of the prim. order, is equivalent to its *first term*, *plus* the *sum* of $(n-1)$ terms of the 1st order of diff.; but the said *sum* may be obtained by the *formula* in Art. 60, which (by Arts. 45 and 46,) is a general expression for any series.

To obtain the required *sum*, the said *formula*, (Art. 60,) is applied to the 1st order of diff., as though it was the *prim. order* to the following differences; and further, the *ascending order* of the 1st diff. is in number, a *unit*.

Now by Art. 52, Mz, when reckoned from the 1st diff., (as its prim. order,) is of an ascending order a unit less, than when reckoned from the *prim. order;* that is, its *ascending order* is $(z-1,)$ and it is $(z-2)$ units higher than the 1st diff.; also Mz contains $(z-1)$ terms less than the *prim. order*, and of course $(z-2)$ terms less than the 1st diff., or $(z-2)$ less than $(n-1)$; i. e. it contains $n-(z-1)$ terms.

Hence, in making the requisite application of said formula to the 1st order of diff., we must substitute in that formula $(n-1$ for n; and $(z-2)$ for $(z-1,)$ or, which is the same, $(z-1)$ for z: To this expression, add the first term of the *prim. order*, and we shall have the value of T, which is

A GENERAL FORMULA,

Expressing the n*th term of the Prim. Order of Series.*

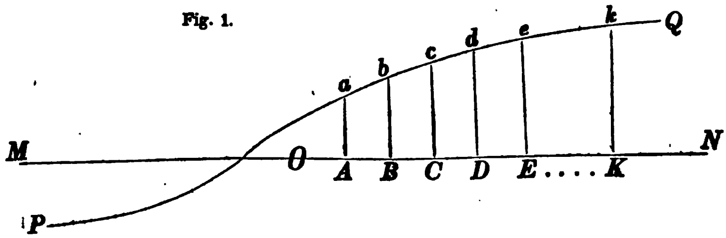
$$T=A+\mathbf{a}\,.\frac{n-1}{1}+\mathbf{b}\,.\frac{(n-1)(n-2)}{1\times 2}+\mathbf{c}\,.\frac{(n-1)(n-2)}{1\times 2}+\ldots\ldots$$

$$\ldots\ldots+\mathbf{w}\,.\frac{(n-1)(n-2)(n-3)\,.\,.\,.\,.\,.\,.[n-(z-1)]}{1\times 2\times 3\times\ldots\ldots\times(z-1)}+Q_{z+1}$$

Schol. **The two Schols. in Art. 60, are also equally applicable to the above formula.**

62. *To show in what manner the Formula in Art.* 61, *may be considered the Equation of a Geometrical line.*

Fig. 1.



In accordance with "Analytical Geometry," let the indefinite right line MN (the *axis of Abscissas* in Fig. 1,) be divided into any number of equal parts, (each part being taken for the UNIT of measure,) and, at each point of division, let an indefinite right line be drawn perpendicular to MN; moreover, let the indefinite line PQ be drawn cutting the *said perpendiculars*, so that their respective *parts*, limited by, and contained between the two lines MN and PQ, shall, (in comparison with the *unit* measure,) when taken successively from O towards N, (O being one of the supposed points of division) represent the successive terms of any order of series whatever.

Now let O be the *origin of Abscissas*, then the *distance* from O, (on MN, in *unit* measures) to the foot of any one of the said perpendiculars, is the *Abscissa* of that perpendicular, which is termed the *Ordinate* of the *Abscissa*. In like manner, each of the *said* perpendiculars is an *Ordinate*, having its corresponding *Abscissa*.

According to the foregoing supposition, if $OK = n$, (that is n times the *unit measure*,) Kk will be the nth Ordinate, and will contain the *unit measure* as many times as the nth term of the supposed series contains units. In like manner, if $OA = 1$, Aa represents the *first term* of the series, and so on for any other Abscissa and its Ordinate.

In the formula, (Art. 61,) for n, substitute x, and for T, y; (x and y denoting any *Abscissa* and *Ordinate* in the equation of a line;) then, that formula becomes the equation of *some line*, wich will cut the successive supposed ordinates in the same points that the line PQ is supposed to cut them; for when x becomes an integer value equal to n, (see Art. 8,) it differs not from n in the formula, and of course y will not differ from the nth ordinate, as it will represent T, in the formula. But the *said line*, of which x and y are the co-ordinates, does *more* than simply to cut the supposed ordinates in Fig. 1; because—1. It limits y, corresponding to *any* value of x, whether integer or not; negative or positive.—2. It cuts MN, at a point, whose distance from O is equal to that value of x, which causes y to be equal zero.—3. It is below the line MN, wherever the value of x gives a negative y.

Moreover, when x is taken from O towards N, it is positive, but from O towards M it is negative.

Hence, from the above it is obvious,—1. That the *supposed* line PQ, in Fig. 1, may be considered the line, of which x and y are the co-ordinates.—2. That when x receives the successive integer values of n, x becomes identical with n, and y with T, and, as every value of x (by the foregoing) has a corresponding value to y, so, for the same reasons, every value of n, whether an integer or a fraction, negative or positive, has a corresponding value, T.—3. All that is said in the foregoing, respecting the formula in Art. 61, is equally applicable to those formulæ in Arts. 58 and 38.

REMARKS. 1. The foregoing is no doubt a sufficient explanation of the nature of taking n (the number of terms) either as a *whole* or *fractional* number, negative or positive. Also, what is meant by a negative or positive term T'.

2. Any order of series whose $Q_{x+1} = 0$, (see Art. 59,) may be reduced to the equation of a line, by substituting in the resulting formula, x for n, and y for T.

SECTION II.

To show the manner of applying the General Formulæ (in Arts. 60 and 61,) *in order to obtain other Formulæ, which shall be applicable to sub-species of series.*

In the following, let S, T, and n, denote the same as they do in the general formulæ, (Arts. 60 and 61.)

63. ARITHMETICAL PROGRESSION

Is a *species of series*, such that the difference between every two consecutive terms, is equal to a constant quantity.

Agreeably to this definition, let A be the first term; T the nth term; n the number of terms; d the constant difference; and S the sum of n terms. Then, from these five data, we obtain the two following formulæ, which are sufficient to solve any problem in this species of series, when *three* of the *five* data are given. Thus, we have

Primitive Order,	A,		B,		C,		D,		E,		F,
1st Differences,		d		d		d		d		d	

Hence,

By Art. 61, $T = A + (n - 1)\,d$ (1)

By Art. 60, $S = nA + \frac{n(n-1)}{1\ 2}d = \frac{2A + (n-1)d}{2} \cdot n$

By substituting the value of T,

we have $$S = \frac{A+T}{2} \cdot n \qquad (2)$$

64. PROGRESSIONS BY POWERS.

Let the successive terms of any given Series be raised to a given Power.

CASE I.

To find the sum of n *terms of the natural arithmetical numbers cubed,* we proceed as follows:

Prim. Order,	1^3,	2^3,	3^3,	4^3,	5^3,
or thus,	1,	8,	27,	64,	125,
1st Diff.	7,	19,	37,	61,	
2d Diff.	12,	18,	24,		
3d Diff.	6,	6,			
	0,				

Then by Art. 60, we have

$$S = n + 7 \cdot \frac{n(n-1)}{1\ 2} + 12 \cdot \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} + 6 \cdot \frac{n(n-1)(n-2)(n-3}{1 \cdot 2 \cdot 3 \cdot 4}$$

By reduction, $$S = \frac{n^2(n+1)^2}{4} \quad \text{or,} \quad S = \frac{[n(n+1)]^2}{1 \cdot 2}$$

CASE II.

To find the sum of n *terms of the fourth Powers of the natural arithmetical numbers;*

Prim. Order,	1^4	2^4	3^4	4^4	5^4	6^4	7^4
or thus,	1,	16,	81,	256,	625,	1296,	2401,
1st Diff.	15,	65,	175,	369,	671,	1105,	
2d Diff.	50,	110,	194,	302,	434,		
3d Diff.	60,	84,	108,	132,			
4th Diff.	24,	24,	24,				
	0,	0,					

Then by Art. 60, we have,

$$S = n + 15 \cdot \frac{n(n-1)}{1 \cdot 2} + 50 \cdot \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} +$$

$$60 \cdot \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} + 24 \cdot \frac{n(n-1)(n-2)(n-3).}{1 \cdot 2 \cdot 3 \cdot 4}$$

And by reduction we obtain

$$S = \frac{n\,(6\,n^4 + 15\,n^3 + 10\,n^2 - 1)}{30},$$

or,
$$S = \frac{n\,(n+1)\,(2\,n+1)\,(3\,n^2+3\,n-1)}{30}.$$

CASE III.

To find the sum of n *terms of the fifth Powers of the natural arithmetical numbers:*

$1^5,\ 2^5,\ 3^5,\ 4^5,\ 5^5,\ 6^5,\ \ldots\ldots\ldots\ldots\ n^5.$

$$S = \frac{(n^2+n)^2(\,2n^2+2n-1)}{12};\ \text{Answer.}$$

Scholium. The sum of n terms of any order of powers, may be obtained, by proceeding according to the foregoing *Cases.*—By the Formula in Art. 61, we might have obtained the value of T, but this was unnecessary, for in all cases, it is equal to the nth term raised to the given power; as in Case III. $T=n^5$.

ON PILING BALLS.

65. Cannon balls are piled in three different forms;—1. In form of a pyramid, with a triangular base, and one ball at the apex;—2. In form of a pyramid having a square base, and one ball at the apex;—3. In a rectangular pile, having a rectangle for its base, and a row of balls for its apex.

Now in *each* of the three kinds of piles, let the number of balls, in the successive tiers, from the top downwards, be the successive terms of a series of numbers, and let n represent the number of tiers, in each pile, and r the number of balls in the *row* which forms the apex of the *rectangular pile:* S will then denote the number of balls in the respective piles.

The successive numbers, corresponding to five successive tiers from the top downwards, in each of the three piles respectively, are,

For Triangular piles,	1,	3,	6,	10,	15,	(1)
For Square base piles,	1,	4,	9,	16,	25,	(2)
For Rectangular piles,	r,	$2r+2$,	$3r+6$,	$4r+12$,	$5r+20$	(3)

Hence, according to Art. 59, we obtain the following general expressions:

CASE I.

For a Triangular pile of balls,

Primitive Order,	1,		3,		6,		10,		15,
1st Diff.		2,		3,		4,		5,	
2d Diff.			1,		1,		1,		
				0,		0,			

Hence,

By Art. 60, $S = n + 2 \cdot \frac{n(n-1)}{1 \cdot 2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 2}$

By Reduction, $S = \frac{n(n+1)(n+2)}{6}$

CASE II.

For a Square pile of balls,

Primitive Order,	1,		4,		9,		16,		25,
1st Diff.		3,		5,		7,		9,	
2d Diff.			2,		2,		2,		
				0,		0,			

By Art, 60, $S = n + 3 \cdot \frac{n(n-1)}{1 \cdot 2} + 2 \cdot \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$

By Reducing, $S = \frac{n(n+1)(2n+1)}{6}$

CASE III.

For the Rectangular pile,

Prim. Order,	r,		$2r+2$,		$3r+6$,		$4r+12$,		$5r+20$
1st Diff.		$r+2$,		$r+4$,		$r+6$,		$r+8$,	
2d Diff.			2,		2,		2,		
				0,		0,			

By Art. 60, $S = rn + (r+2) \cdot \frac{n(n-1)}{1 \cdot 2} + 2 \cdot \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$

By Reduction, $S = \frac{n(n+1)(2n+3r-2)}{6}$

SERIES IN GENERAL.

66. Let it be required to find the 31st term, and also the sum of 31 terms of the series. 1, 3, 8, 16, 27, &c.

1st. Make the Rules, thus,

Primitive Order,	1,		3,		8,		16,
1st Diff.		2,		5,		8,	
2d Diff.			3,		3,		
				0.			

By Art. 61, $T = 1 + 2 \cdot \frac{n-1}{1} + 3 \cdot \frac{(n-1)(n-2)}{1 \cdot 2}$, and

By Art. 60, $S = n + 2 \cdot \frac{n(n-1)}{1 \cdot 2} + 3 \cdot \frac{n(n-1)\ (-2)}{1 \cdot 2 \cdot 3}$

By reducing the above, we obtain

$$T = \frac{n(3n-5)}{2} + 2, \text{ and } S = \frac{n(n-1)+2}{2} \times n$$

2d. Let n equal 31; then, by the foregoing formulæ, we obtain the required answers, thus;

$$T = \frac{31\ (3 \times 31 - 5)}{2} + 2 = 1366, \text{ the 31st term; and}$$

$$S = \frac{31\ (31-1)+2}{2} \times 31 = 14446, \text{the sum of 31 terms.}$$

In like manner n may be taken for any number. (Art. 8.)

Scholium. In a similar manner any numerical series may be treated, viz:—1. Make the requisite *Rules*, and,—2. Solve the *problems* by said rules.

BY APPROXIMATIONS.

67. *Example* 1. The Natural Sine of 21° 34′ is, ‘36758; of 21° 35′, ‘36785; of 21° 36′, ‘36812; and of 21° 37′, ‘36839. It is required to find the Nat. Sine of 21° 35′ 47″.

Now 21° 34′ is the first term of the series, and 21° 35′, the second; and as 21°35′47″ is $\frac{47}{60}$ of the distance between the second and third terms, it is therefore the $2\frac{47}{60}$th term in the series.

Hence, by taking the Differences, we have

	Prim. Order.	1st Differences.	
Sine 21° 34′ =	‘36758		
		‘00027	
“ 21° 35′ =	‘36785		0
		‘00027	
“ 21° 36′ =	‘36812		0
		‘00027	
“ 21° 37′ =	‘36839		

By Art. 61, $T = {}^{\backprime}36758 + {}^{\backprime}00027 \cdot \frac{n-1}{1}$. (Here $Q_{x+1} = 0$)

By substituting $2\frac{47}{60}$ for n, we have

$$T = {}^{\backprime}36758 + {}^{\backprime}00026\ (2\tfrac{47}{60} - 1) = {}^{\backprime}36806,$$

That is, Sine 21° 35′ 47″ = ‘36806.

Example 2. The Logarithm of 4902 is 3‘690373; of 4904, 3‘690550; of 4906, 3‘690728; and of 4908, 3‘690905. It is required to find the log. of 4905,

Now, 4902, is the first term, and 4904, the second, and, as 4905 is half way between the 2d and 3d terms, it is itself the $2\frac{1}{2}$ term in the series. Hence,

	Prim. Order.	*1st Diff.*	*2d Diff.*
Log. 4902	= 3‘690373		
		‘000177	
" 4904	= 3‘690550		+‘000001
		‘000178	
" 4906	= 3‘690728		—‘000001
		‘000177	
" 4908	= 3‘690905		

By Art. 61, (Making $Q_{x+1} = 0$), we have

$$T = 3‘699373 + (n-1) \times ‘000177 + \frac{(n-1)(n-2)}{1\ .\ 2} \times ‘000001$$

By substituting $2\frac{1}{2}$ for n and reducing, we have

Log. 4905 = 3‘690639.

The foregoing applications are deemed sufficient to illustrate the principle.

CHAPTER III.

INVOLUTION AND EVOLUTION OF BINOMIAL QUANTITIES.

70. Let a and x be *any* two numerical quantities whatever, then their *sum* or *difference*, as $a+x$, or $a-x$, will be the *Binomial quantity*, the *Involution* and *Evolution* of which, are the subjects for discussion in this chapter.

INVOLUTION.

SECTION I.

This section relates to the law regulating the powers of a *and* x.

71. Let it be supposed,

1st. That a and x represent any two numerical quantities whatever.

2d. That several Algebraic terms are arranged in the direction of a right line from left to right, and that each term in the rank, besides its co-efficient, is composed of *two factors*, which are a and x; denoted by their respective powers. Also, let the co-efficients of the respective terms, in said rank, be represented by capital letters, in alphabetical order from left to right.—[Rank 1, Art. 74.]

3d. That the first term, on the left of said rank, be denoted by $Aa^m x^0$, and the whole number of terms in the rank, by $m+1$, (m denoting any positive integer number; see Art. 8,) and furthermore, that, respecting *any two consecutive terms* of the rank, in the term to the *left*, the power of a is a unit *greater*, and the power of x a unit *less*, than the respective powers of a and x in the term next to the *right*.

According to the above supposition, in the successive terms of said rank, (taken from *left* to *right*,) the corresponding powers of a constitute a *Decreasing Arithmetical Series*, [see Art. 63,] whose first term is m, No. of terms $m+1$, and common difference 1, (or -1); and the corresponding powers of x constitute an *Increasing Arithmetical Series*, whose first term is 0, No. of terms, $m+1$, and common difference 1.

Now in any term of the supposed rank, as the kth term, let T denote the power of a, and t the power of x; then (by Art. 63, Formula (1)) $T=m-(k-1)=m-k+1$, and

$t = 0 + (k - 1) = k - 1$; and $T + t = m$. Moreover, when $k = m + 1$, $T = m - (m + 1 - 1) = 0$, and $t = 0 + (m + 1 - 1) = m$.

71 (a). Hence, relative to the supposed rank of terms—1. The greatest power of a is the same as that of x, and the least power of a, and of x equals zero.—2. The *sum* of the powers of a and x in *any* term of the rank, equals the highest power of a or x.—3. The number of terms in the rank is one greater than the highest power of a or x.—The successive powers (from left to right) of a, constitute a *Decreasing*, and of x, an *Increasing* Arithmetical Series; and in each series, the *greatest* term is a unit less than the number of terms in the rank, the least term is zero, and the common difference is 1.

LEMMA.

72. *The common difference, pertaining to an Arithmetical Series, will not be changed, when each term of the series is augmented by the same value;—*

For let any two consecutive terms of the series, be denoted by y and z, and their difference by $y - z$; then, if each term be augmented by w, we have $y + w$, and $z + w$, whose difference is $y - z$, the same as before. In like manner the same may be proved of every two consecutive terms. Q. E. D.

(72 a). *Corol.* From the foregoing, it is evident, that if the supposed rank, in Art. 71, be multiplied by a and by x, the successive powers of a and of x, will be respectively augmented by 1, and in each series of powers, the common difference will remain 1, and the number of terms $m + 1$; but *now* the greatest term will be $m + 1$, and the least term 1.

LEMMA.

73. In the supposed rank of Series, (Art. 71), let any two consecutive terms be taken, as the kth and $(k + 1)$th terms. Then, *if* the $(k + 1)$th term be multiplied by a, and the kth term by x; the respective powers of a and of x, in each product will be equal, and therefore the two products may be added, by prefixing the sum of their co-efficients, to the common factors of a and x.

For, according to Art. 71, the kth term is $Ka^{m-k+1} x^{k-1}$, and the $(k + 1)$th term is $La^{m-k} x^{k}$. By performing the *conditional* multiplication, the $(k + 1)$th term becomes $La^{m-k+1} x^{k}$, and the kth term, $Ka^{m-k+1} x^{k}$; consequently, their sum is $(L + K)a^{m-k+1} x^{k}$. Moreover, the *sum* of the powers of a and x, in the *resulting sum*, is $m - k + 1 + k = m + 1$.

Let rank (1) be the supposed *rank of terms*, described in Art. 71, in whose kth term, the power of a is $m-k+1$, and the power of x, $k-1$; and in whose $(k+1)$th term the power of a is $m-k$, and of x is k.

Now, let rank (1) be multiplied by $a+x$; and let rank (2) be the product of a *into* rank (1), and rank (3), the product of x *into* rank (1); by this means the successive powers of a, and of x, will be respectively augmented by 1, according to Art. 72, *cor.* Moreover, let the successive terms in rank (3), be, respectively, removed one place towards the right, so that the kth term of rank (3), [which is the kth term of rank (1) *into* x,] shall be directly under the $(k+1)$th term of rank (2), [which is the $(k+1)$th term of rank (1) into a]; then (Art. 73,) these two terms may be added together, and their sum will be the $(k+1)$th term in rank (4). In like manner each term in rank (2) may be added to the term directly under it, in rank (3), and thus form a corresponding term in rank (4), for k may be taken successively for all integer numbers from 1, to m. (Art. 8.)

Hence, rank (4) is the product of rank (1) into $a+x$, and it possesses the following properties:—1. By supposition, the last term in rank (3), is moved one place to the right of the last term in rank (2), which rank has as many terms as rank (1), that is, $(m+1)$ terms; consequently rank (4) must contain one term more than rank (2) or rank (1); that is, it contains $(m+2)$ terms.—2. In any term, as the $(k+1)$th term of rank (4), the powers of x, and a, are the same, as their respective powers, in the $(k+1)$th term of rank (2), and in the kth term of rank (3); [k being taken according to Art. 8.] Moreover, the first term of rank (4), containing x^0, is one place

74. *The process of multiplying Rank* 1, *into* a + x.

$$\begin{array}{llllllllll}
Aa^mx^0 & +Ba^{m-1}x^1 & +Ca^{m-2}x^2 + \dots & +Ka^{m-k+1}x^{k-1} & +La^{m-k}x^k & + \dots & +Ua^1x^{m-1} & +Va^0x^m & & (1) \\
a & +x & & & & & & & & \\
\hline
Aa^{m+1}x^0 & +Ba^mx^1 & +Ca^{m-1}x^2 + \dots & +Ka^{m-k+2}x^{k-1} & +La^{m-k+1}x^k & + \dots & +Ua^2x^{m-1} & +Va^1x^m & & (2) \\
 & Aa^mx^1 & +Ba^{m-1}x^2 + \dots & +Ja^{m-k+2}x^{k-1} & +Ka^{m-k+1}x^k & + \dots & +Ta^2x^{m-1} & +Ua^1x^m & +Va^0x^{m+1} & (3) \\
\hline
Aa^{m+1}x^0 & +\left.\begin{matrix}B\\A\end{matrix}\right| a^mx^1 & +\left.\begin{matrix}C\\B\end{matrix}\right| a^{m-1}x + \dots & +\left.\begin{matrix}K\\J\end{matrix}\right| a^{m-k+2}x^{k-1} & +\left.\begin{matrix}L\\K\end{matrix}\right| a^{m-k+1}x^k & + \dots & +\left.\begin{matrix}U\\T\end{matrix}\right| a^2x^{m-1} & +\left.\begin{matrix}V\\U\end{matrix}\right| a^1x^m & +Va^0x^{m+1} & (4)
\end{array}$$

to the left of the first term of rank (3); also the last term of rank (4) containing a^0, is one place to the right of the last term in rank (2); consequently, (Art. 72*a*), in rank (4), the successive powers of a constitute a *Decreasing*, and the successive powers of x, an *Increasing*, Arithmetical Series; in each of which, the greatest term is $m+1$, the least term 0, the number of terms $m+2$, and the common difference 1.—3. In any term of rank (4), as the $(k+1)$th term, the sum of the powers of a, and x, equals $m-k+1+k=m+1$; that is, is equal to the highest power of a, or x; or to a unit less than the number of terms in the rank.—4. The co-efficients of the *first* and *last* terms of rank (4), are the same as those of the *first* and *last* terms in rank (1.) Hence,

Rank (4) *possesses the same properties attributed to rank* (1) in Art. 71a, *or the supposed rank* in Art. 71.

75. According to Art. 74, it is evident, that if a rank of terms, which is embraced by the supposition, in Art. 71, be multiplied *into* $a+x$, the *product* thus arising, will, also, be a rank of terms embraced in the same *supposition*; consequently, *this product* may be multiplied into $a+x$, producing a like result. In like manner, $a+x$ may be successively taken for a multiplier $n-1$ times, and the last result will be embraced in the general supposition. But, $a+x$ taken for a successive multiplier $n-1$ times, is the same as multiplying the *supposed rank* into $(a+x)^{n-1}$.

(75*a*). Again, (Art. 74,) each of the *successive products*, arising by taking $a+x$ for a multiplier, contains one term more than its multiplicand; consequently, the $(n-1)$th product will contain $n-1$ terms more than the supposed rank, which has $m+1$ terms by sup. Hence, the ultimate product, produced by multiplying the supposed rank into $(a+x)^{n-1}$, will contain $m+n$ terms, and consequently, (Art. 71 *a*,) the greatest power of a, and of x, will be a unit less than $m+n$, or $m+n-1$.

Now, let the supposed rank in Art. 71, be such that m, A, and B, each equals 1; then we have for the supposed rank, ax^0+a^0x, which answers to the supposition. But as x^0, and a^0, are each equal to 1, $ax^0+a^0x=a+x$; therefore, $(ax^0+a^0x)\times(a+x)^{n-1}=(a+x)^n$. Hence,

According to the foregoing, (without taking co-efficients into the account,) $(a+x)^n$ equals a rank of series, consisting of $n+1$ terms, (m being equal to 1,) whose first term is a^n, $(=a^n x^0,)$ and last term, x^n, $(=a^0 x^n)$. Moreover, the suc-

cessive powers of a, constitute a *Decreasing*, and the successive powers of x, an *Increasing* Arithmetical Series, whose common difference is 1. Also the co-efficients of the *first* and last terms will be 1. [See Art. 74, No. 4.] Furthermore, a is in every term except the last, and x in every term except the first, for a^0, and x^0, each equals 1.

SECTION II.

This section relates to the co-efficients arising by expanding $(a+x)^n$.

76. In passing from $(a+x)^1$ to $(a+x)^n$, by *successive products*, (Art. 75*a*,) each *product* has one more term in its rank, than the next preceding *product*, and also one more term than the No. denoting the power of $(a+x)$, from which the said product is derived (Art. 75); therefore, in the *successive products*, the $(k+1)$th term first makes its appearance, in the development of $(a+x)^k$, and the co-efficient of that *term* is 1, (Art. 74, No. 4.)

Again, let *ranks* (1) and (4) (Art. 74) be any two *consecutive products*, taken from the said *successive products*; then (Art. 74) it has been proved, that the same relations, which exist between the $(k+1)$th co-efficient of rank (4), and that of rank (1), also exist between all their corresponding co-efficients. In like manner, the relations which exist between *any* two consecutive successive products, as ranks (1) and (4), also exist between *every* two consecutive products.

Now in rank (1) (by supposition), L denotes all the terms which constitute the $(k+1)$th co-efficient, and K, those which constitute the kth co-efficient; but it has been proved that L augmented by the new term K, constitutes the co-efficient of the $(k+1)$th term in rank (4). In like manner, we may pass through the said *successive products*, until we arrive at that derived from $(a+x)^n$, and at each successive step, L is augmented by a new term K, which, at last, is the *sum* of all the terms of the kth co-efficient derived from $(a+x)^{n-1}$, or the next preceding product; and, therefore, in the development of $(a+x)^n$, *any co-efficient*, as the $(k+1)$th, is the first *Ascending order of series*, arising (Art. 10,) from the terms in the kth co-efficient, (next to the left,) as its *basis*.

Further, according to the preceding, the $(k+1)$th *term* (and also its co-efficient, equal 1) first makes its appearance in the development of $(a+x)^k$; but in passing to $(a+x)^n$, the multiplier $a+x$, has been successively taken $n-k$ times, and at each time said co-efficient has been augmented by a new term; consequently, the whole number of terms in the $(k+1)$th co-efficient, derived from $(a+x)^n$, is $n-k+1$. Hence,

77. In the development of $(a+x)^n$,

1. The co-efficient of the second term contains n units, for as was proved above, the first term in this co-efficient is 1; and (Art. 74, (4)) the first terms of each product has for its co-efficient 1; consequently, every term by which this co-efficient is augmented is 1; that is, when $k+1$ equals 2, or denotes the second co-efficient, then the number of the terms (according to the above) equals $n-k+1$, or n: therefore n is the second co-efficient.

2. The co-efficient of any term, except the first, (Art. 74, No. 4,) contains one term more than the co-efficient of the term next to the right. This is evident, for in the expression, $n+1-k$, the number of terms, (Art. 76,) k may be successively taken for all integer values from 1 to n. Let m denote any integer value not less than 2, nor greater than $n+1$, then it is evident from the foregoing that the mth co-efficient contains $m-2$ terms less than the second co-efficient.

3. According to what is proved above, each co-efficient by Art. 10, produces the co-efficient next to the right; hence, the second co-efficient (consisting of units) is the *basis* of an *Ascending System of Series*, and the successive co-efficients, after the second, are the successive ascending orders, arising (Art. 10,) from this basis. Consequently, the mth co-efficient will be an *Ascending Order of Series*, $m-2$ units higher than the second co-efficient.

78. The supposition being the same, concerning the development of $(a+x)^n$, as in Art. 77, it is evident from that article;—1. That the second co-efficient is the *sum* of n units, and, from the nature of its composition, may be taken as a *series* of n units, which (series) may also be the basis (or *first order*) of an *Ascending System of Series*, whose successive *orders*, (arising according to Art. 10) are (commencing at the second co-efficient) the successive co-efficients in the said *development*.—2. In any co-efficient, after the second, as the mth, the number of terms is as many units less than n, (the number of terms in the

second co-efficient,) as the number, denoting its *ascending order*, is units higher than 1, the basis being considered the 1st order.

Consequently, each *co-efficient*, (commencing at the second,) in accordance with the *number of its terms*, and *number of its Ascending order*, may be expressed by the formula in Art. $39\frac{1}{2}$. But the factor a, in said formula, at each application, is made 1, as the terms int he *basis* of the Ascending System of Series consists of units. Hence,

In the development of $(a+x)^n$,

The co-efficient of		the 1st term		$= 1$,	
"	"	"	2d	"	$= n$,
"	"	"	3d	"	$= \frac{n(n-1)}{1 \;.\; 2}$
"	"	"	4th	"	$= \frac{n(n-1)(n-2)}{1 \;.\; 2 \;.\; 3}$
"	"	"	5th	"	$= \frac{n(n-1)(n-2)(n-3)}{1 \;.\; 2 \;.\; 3 \;.\; 4}$
.	.	.	.		
.	.	.	.		
"	"	"	$n+1$th	"	$= 1$, [see Art. 74, (4).]

Note. The $(n+1)$th co-efficient contains n factors in both numerator and denominator, in each of which the greatest factor is n, and leastfactor 1: hence, the co-efficient will be 1, as stated above. But should we endeavor to find the $(n+2)$th term, it would be zero, for in its co-efficient $n-n$, or 0, would be a factor, and this would be the case respecting any imagined term to the right of the $(n+1)$th term.

SECTION III.

Binomial Formulæ and their Application.

79. As, concerning negative quantities, even powers produce *positive*, and uneven powers *negative* quantities, it is evident that what is said of $(a+x)^n$, is equally applicable to the development of $(a-x)^n$. Consequently, according to what has been said in Arts. 75 and 78, we are enabled to develop any Binomial

quantity, which is under a positive integer power. Hence, we have

80. THE BINOMIAL FORMULÆ.

(a)

$$(a+x)^n = a^n + \frac{n}{1}a^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3 + \&c.$$

(b)

$$(a-x)^n = a^n - \frac{n}{1}a^{n-1}x + \frac{n(n-1)}{1.2}a^{n-2}x^2 - \frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3 + \&c.$$

By inspecting the above formulæ, we easily discover the general relations which exist between any two consecutive terms in the *development.*

Hence by denoting, in any *Binomial expression*, the first term by a, and the second by x, we have for the sake of convenience in aiding the memory, the following

81. GENERAL RULE,

For expanding Binomial Quantities.

1st. *Raise* a *to the given power of the Binomial.*

2d. *To derive any term, after the first in the rank, multiply the term already found, into the power of* a *in that term, and then into* x; *after which, divide the product by the number denoting the place of said term in the rank, and then by* a, *and the quotient will be the next term in the rank towards the right: thus, by successive steps, all the terms in the development may be derived.*

Examples.

1. $$(a+x)^4 = a^4 + 4a^3x + \frac{4\times 3}{1.2}a^2x^2 + \frac{4.3.2}{1.2.3}ax^3 + \frac{4.3.2.1}{1.2.3.4}x^4$$
$$= a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$$

In like manner,

2. $(t-z)^5 = t^5 - 5t^4z + 10t^3z^2 - 10t^2z^3 + 5tz^4 - z^5$

3. $(3a^2 + 5d)^3 = (3a^2)^3 + 3(3a^2)^2(5d) + 3(3a^2)(5d)^2 + (5d)^3$,
By reducing, $= 27a^6 + 135a^4d + 225a^2d^2 + 125d^3$

4. $a+b-c$ equals $a+(b-c)$; therefore,
$$[a+(b-c)]^3 = a^3 + 3a^2(b-c)^2 + (b-c)^3.$$

By reducing,

$$=a^3+3a^2b-3a^2c+3a(b_2-2bc+c^2)+b^3-3b^2c+3bc^2-c^3,$$

or, $$=a^3+3a^2b-3a^2c+3ab^2-6abc+3ac^2+b^3-3b^2c+{}^3bc^2-c^3.$$

It is supposed that the foregoing examples are sufficient to illustrate, and show the application of, the Binomial Formulæ.

EVOLUTION,

Or the Expansion of a Binomial Quantity, which has a fractional power; as, $(a \pm x)^{\frac{m}{n}}$.

SECTION I.

81. Let n denote any *finite* number of units, and m any number of units, either *finite* or *infinite*.

Then, according to Art. 80,

$$(a+x)^m = a^m + \frac{m}{1}a^{m-1}x + \frac{m(m-1)}{1\ .\ 2}a^{m-2}x^2 + \text{\&c.} \qquad (1)$$

And in like manner,

$$(a+x)^{mn} = a^{mn} + \frac{mn}{1}a^{mn-1}x + \frac{mn(mn-1)}{1\ .\ 2}a^{mn-2}x^2 + \text{\&c.} \quad (2)$$

81½. *Remark.* The second member of equation (1) may be considered as a series of the most general character, either *finite* or *infinite*, whose several terms succeed each other according to the *Law* of the *Binomial expansion*.

82. *Explanation.* When the *index* of a *Binomial quantity* enters the expanded series, it may be termed the INDEX *of the Series;* and hence, throughout the development of $(a+x)^n$, as in Art. 80, both in the successive powers of a, and also in the successive co-efficients, n is the INDEX *of the Series*.

83. Let the second member of equation (1)(Art. 81,) be denoted by A, and that of equation (2), by B; then we have from equation (1) (by raising *equals* to like powers) $(a+x)^{mn} = A^n$; but by equation (2), $(a+x)^{mn} = B$; hence, $A^n = B$; that is, if the second member of equation (1) were actually raised to the nth power, and reduced, the *result* would take the form of, and be identical with, the second member of equation (2).—Hence, generally,—

Any series, *finite* or *infinite*, whose terms succeed each other according to the Law of the *Binomial expansion*, may be raised to a *given power*, simply by multiplying that number, denoting the *given power*, into all the terms throughout the *expansion*, which answers to the Index *of the Series*, (Art. 82.)

OF ROOTS IN GENERAL.

84. In order to extract *any* root of a quantity, as the nth root, it is necessary to find a quantity, which, by involving it to the nth power, will re-produce the original quantity. Therefore, any quantity, which *has* an nth *root*, may be considered as the *result* of the involution of that root to the nth power.

85. Let it be required to find the nth root of the second member of equation (2) (Art. 81). Now by dividing the *Index of the Series* by n, we obtain the second member of equation (1), which is the required root, according to Art. 84; for, by involving this root to the nth power (according to Art. 83,) the second member of equation (2) is re-produced.

In like manner, let it be required to find the nth root of any series, *finite* or *infinite*, [for instance, the second member of equation (1) Art. 81,] whose terms succeed each other, according to the *law* set forth in the Formulæ, Art. 80.

Now by dividing by n, throughout the series, all those numbers (m's) which answer to the *Index* of the series, (Art. 82,) [which process is evidently the same as substituting in the series, $\frac{m}{n}$ for m;] we obtain the required nth root; for, were we to involve *this root* to the nth power, (Art. 83), by multiplying each $\frac{m}{n}$ into n; each $\frac{m}{n}$ would become m, and also, the second member of equation 1, would be re-produced. Hence, the *obtained series* is the required nth *root*, according to Art. 84.

86. Now, as the nth roots of *equals* are also *equals*, let the nth root of both members of equation (1) (Art. 80,) be taken.—By familiar principles in Algebra, the nth root of the first member, $(a+x)^m$, becomes $(a+x)^{\frac{m}{n}}$; and, according to Art. 85, the nth root of the second member is the same as would be produced by substituting throughout the second member, $\frac{m}{n}$ for m. Hence, by using the double sign ($\pm$) to avoid re-writing two formulæ as in Art. 80, we have the following

GENERAL FORMULA,

For fractional powers:

$$(a\pm x)^{\frac{m}{n}}=a^{\frac{m}{n}}\pm\frac{m}{n}a^{\frac{m}{n}-1}x+\frac{\frac{m}{n}\left(\frac{m}{n}-1\right)}{1\,.\,2}a^{\frac{m}{n}-2}x^2\pm\frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{1\,.\,2\,.\,3}a^{\frac{m}{n}-3}x^3+$$

Scholium. The successive co-efficients in the above formula, are denoted by the signs $+$ and $-$, according to the general *law* of expansion; but the essential sign of each co-efficient depends upon the signs of the respective factors, that compose the said co-efficient. For any factor, as $\frac{m}{n} - d$, which may enter a co-efficient, is always negative, when $\frac{m}{n}$ is a proper fraction, and d a positive integer number.

87. Whether the Binomial power be an *integer* or a *fraction*, provided it be *positive*, we see, by Arts. 80 and 86, that the series will be expanded according to the same *law*. Our next inquiry will relate to *negative powers*.

Let c denote either an *integer* number, or a *fraction*, and z any positive variable number whatever. Then $(a+x)^{z-c}$ equals a series expanded according to Art. 80, which will be the same as the formula in Art. 80, when $z-c$ is substituted throughout the series, for n. Now this series will evidently be true for any value of z, even when it becomes zero; but when $z=0$, $(a+x)^{z-c}$ becomes $(a+x)^{-c}$, and in the formula (Art. 80,) $-c$ takes the place of n.

Hence, whether the Binomial power be an *integer* or a fraction, negative or positive, in every case the Binomial Quantity will be expanded according to the Formulæ in Arts. 80 and 86.

88. According to Art. 75, if $n > 1$, no nth power can produce a *Binomial quantity:* therefore, no exact nth root can be obtained from a Binomial quantity; for could it be so, this nth root could be raised to the nth power, and re-produce the *Binomial.* Hence, the nth root of a Binomial quantity must be an approximate root, that is, an Infinite Series.

Again, it is evident from Art. 75, that, when a polynomial is raised to the mth power, the *result* will consist of more terms than the said polynomial. Accordingly, were the nth root of a Binomial quantity, (i. e. an infinite series) raised to the mth power, the result would, at the least, be an *infinite series;* that is, $[(a+x)^{\frac{1}{n}}]^m$, or $(a+x)^{\frac{m}{n}}$ would expand to an *Infinite Series.* Nor will any factor in the successive co-efficients, become zero, as in the case when the power is a positive integer number; [see Art. 78, Note] because no positive integer number taken from a proper fraction, which is *negative* or *positive*, can produce zero for the *difference.*

In like manner, it may be shown that, when the Binomial power is a negative integer number, the expansion will be an infinite series.

89. In order to show the *nature* and *application* of the Formula in Art. 86, according to the above principles, the following problems with their solutions, have been inserted.

Example 1.

$$(a+x)^{\frac{3}{5}}=a^{\frac{3}{5}}+\frac{3}{5}a^{\frac{3}{5}-1}x+\frac{\frac{3}{5}(\frac{3}{5}-1)}{1\,.\,2}a^{\frac{3}{5}-2}x^2+\frac{\frac{3}{5}(\frac{3}{5}-1)(\frac{3}{5}-2)}{1\,.\,2\,.\,3}a^{\frac{3}{5}-3}x^3+\ \&c.$$

By reduction,

$$(a+x)^{\frac{3}{5}}=a^{\frac{3}{5}}+\frac{3}{5}a^{-\frac{2}{5}}x-\frac{3\,.\,2}{2\,.\,5\,.\,5}a^{-\frac{7}{5}}x^2+\frac{3\,.\,2\,.\,7}{2.3.5.5.5}a^{-\frac{12}{5}}x^3-\&c.$$

Example 2.

$$(a-x)^{\frac{1}{2}}=a^{\frac{1}{2}}-\frac{1}{2}a^{-\frac{1}{2}}x^1+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1\,.\,2}a^{\frac{1}{2}-2}x^2-\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\,.\,2\,.\,3}a^{\frac{1}{2}-3}x^3+\ \&c.$$

By reduction,

$$(a-x)^{\frac{1}{2}}=a^{\frac{1}{2}}-\frac{1}{2}a^{-\frac{1}{2}}x-\frac{1}{8}a^{-\frac{3}{2}}x^2-\frac{1}{16}a^{-\frac{5}{2}}x^3-\frac{5}{128}a^{-\frac{7}{2}}x^4,\ \&c.$$

Example 3.

$$(a-x)^{-\frac{2}{7}}=a^{-\frac{2}{7}}-\left(\frac{-2}{7}\right)a^{-\frac{2}{7}-1}x+\frac{(-\frac{2}{7})(-\frac{2}{7}-1)}{1\,.\,2}a^{-\frac{2}{7}-2}x^2-\&c.$$

or, " $=a^{-\frac{2}{7}}\left(1+\frac{2}{7}a^{-1}x+\frac{2\,.\,9}{2\,.\,7^2}a^{-2}x^2+\&c.\right)$

or, " $=\frac{1}{a^{\frac{2}{7}}}\left(1+\frac{2\,x}{7\,a}+\frac{9\,x^2}{49\,a^2}+\frac{48\,x^3}{343\,a^3}+\&c.\right)$

NOTE. *The foregoing examples, are doubtless sufficient for what they were designed.*

SECTION II.

90. In accordance with Arts. 87 and 88, the formula in Art. 86, is a *general guide*, by which *any* binomial quantity, whose power is either *fractional* or *integer*, may be expanded. But the object of this section, is to give said formula another *form* so that it may be better adapted to general use.

Every binomial quantity takes one or another of the four following forms, viz: $a+x$, $-a-x$, $a-x$, or $-a+x$, which may

be reduced to $\pm(a+x)$, and $\pm(a-x)$, or thus, $\pm\left((a+a\,.\,\frac{x}{a}\right)$, and $\pm\left(a+a\,.\,\frac{-x}{a}\right)$, and finally to $\pm\left(a+a\,.\,\frac{\pm x}{a}\right)$, or sufficiently thus, $\left(a+a\,.\,\frac{\pm x}{a}\right)$.

In any Binomial Quantity, let the first term be denoted by P, and the quotient of the second term by the first, by Q; also let $\frac{m}{n}$ denote the power of the binomial; then from the foregoing it is evident that every binomial quantity is comprehended in $(P+PQ)^{\frac{m}{n}}$. Now, by expanding this general expression, (Art. 86,) and reducing the power of P, we obtain,

$$(P+PQ)^{\frac{m}{n}}=P^{\frac{m}{n}}+\frac{m}{n}P^{\frac{m}{n}}Q+\frac{\frac{m}{n}(\frac{m}{n}-1)}{1\,.\,2}P^{\frac{m}{n}}Q^2+\frac{\frac{m}{n}(\frac{m}{n}-1)(\frac{m}{n}-2)}{1\,.\,2\,.\,3}P^{\frac{m}{n}}Q^3+$$
$$\frac{\frac{m}{n}(\frac{m}{n}-1)\,(\frac{m}{n}-2)\,(\frac{m}{n}-3)}{1\,.\,2\,.\,3\,.\,4}P^{\frac{m}{n}}Q^4+\ \&c.$$

By farther reduction,

$$(P+PQ)^{\frac{m}{n}}=P^{\frac{m}{n}}+\frac{m}{n}P^{\frac{m}{n}}Q+\frac{m(m-n)}{n\,.\,2n}P^{\frac{m}{n}}Q^2+$$
$$\frac{m\,(m-n)\,(m-2n)}{n\,.\,2n\,.\,3n}P^{\frac{m}{n}}Q^3+\&c. \qquad (1)$$

91. By giving P and Q their original values, and separating the 2d member into two general factors, the equation (1) is changed to the following

GENERAL FORMULA.

$$(a\pm x)^{\frac{m}{n}}=a^{\frac{m}{n}}\,(1\pm\frac{m}{n}\frac{x}{a}+\frac{m(m-n)x^2}{n\,.\,2n\quad a^2}\pm\frac{m(m-n)(m-2n)x^3}{n\,.\,2n\,.\,3n\quad a^3}+$$
$$\frac{m(m-n)\,(m-2n)(m-3n)x^4}{n\,.\,2n\,.\,3n\,.\,4n\quad a^4}\pm\ \&c.$$

92. By inspecting the above equation (1) it will be seen that, in the second member, any term, as the $(z+1)$th term, may be separated into 3 factors;—1. That factor which the co-efficient of the zth term receives in passing to the $(z+1)$th co-efficient;—2. A factor equal to the zth term;—3. A factor Q.

Hence, by denoting the successive terms in the 2d member of equation (1), by capital letters in alphabetical order, and by

substituting, in any term, for that factor equal to the next preceding term, a corresponding capital letter, we obtain the following

GENERAL FORMULA,

By which any Binomial Quantity of a *negative* or *fractional* power may be expanded.

$$(P+PQ)^{\frac{m}{n}}=P^{\frac{m}{n}}+\frac{m}{n}AQ+\frac{m-n}{2n}BQ+\frac{m-2n}{3n}CQ+\frac{m-3n}{4n}DQ+$$

In order to show the application of the above formula, I will insert the solution of the following

PROBLEM.

Let it be required to expand $\left(a^{-\frac{3}{5}}+x^{\frac{2}{7}}\right)^{-\frac{4}{9}}$;

Then, $P=a^{-\frac{3}{5}}$; $Q=\dfrac{x^{\frac{2}{7}}}{a^{-\frac{3}{5}}}$ or $=a^{\frac{3}{5}}x^{\frac{2}{7}}$; $m=-4$, and $n=9$, or $\frac{m}{n}=-\frac{4}{9}$

Now by substitution,

1st term,

$$P^{\frac{m}{n}}=\left(a^{-\frac{3}{5}}\right)^{-\frac{4}{9}}=+a^{\frac{4}{15}}=A$$

2d term,

$$\frac{m}{n}AQ=-\frac{4}{9}\cdot a^{\frac{4}{15}}\cdot a^{\frac{3}{5}}x^{\frac{2}{7}}=-\frac{4}{9}a^{\frac{13}{15}}x^{\frac{2}{7}}=B$$

3d term,

$$\frac{m-n}{2n}BQ=\frac{-4-9}{18}\cdot\frac{-4}{9}a^{\frac{13}{15}}x^{\frac{2}{7}}\cdot a^{\frac{3}{5}}x^{\frac{2}{7}}=\frac{4.13}{9.18}a^{\frac{22}{15}}x^{\frac{4}{7}}=C$$

4th term,

$$\frac{m-2n}{3n}CQ=\frac{-4-18}{27}\cdot\frac{4.13}{9.18}\cdot a^{\frac{22}{15}}x^{\frac{4}{7}}\cdot a^{\frac{3}{5}}x^{\frac{2}{7}}=-\frac{4.13.22}{9.18.27}\cdot a^{\frac{31}{15}}x^{\frac{6}{7}}=D$$

5th term,

$$\frac{m-3n}{4n}DQ=\frac{-4-27}{36}\cdot\frac{-4.13.22}{9.18.27}a^{\frac{31}{15}}x^{\frac{6}{7}}\cdot a^{\frac{3}{5}}x^{\frac{2}{7}}=\frac{4.13.22.31}{9.18.27.36}\cdot a^{\frac{8}{3}}x^{\frac{8}{7}}=E$$

Hence,

$$\left(a^{-\frac{3}{5}}+x^{\frac{2}{7}}\right)^{-\frac{4}{9}}=a^{\frac{4}{15}}-\frac{4}{9}a^{\frac{13}{15}}x^{\frac{2}{7}}+\frac{4.13}{9.18}\cdot a^{\frac{22}{15}}x^{\frac{4}{7}}-\frac{4.13.22}{9.18.27}a^{\frac{31}{15}}x^{\frac{6}{7}}+$$

$$\frac{4.13.22.31}{9.18.27.36}a^{\frac{8}{3}}x^{\frac{8}{7}}-\&c.$$

NOTE. *In a similar manner any Binomial Surd may be expanded.*

APPENDIX.

Being thus requested, I present the following solution of a Geometrical problem. The problem is this:

"From a point in an equilateral triangle, the distances to the three angles are, respectively, 20, 29, and 30 rods: Required, the *length of each side*, and the *area* of the triangle."

I would remark, that this problem is only a particular one, embraced in a more general one, in which the said point should be either *within* or *without* the triangle; and its several *distances* from the several angles, should be given in general terms.— Furthermore, this *last* is also but a *particular* problem, embraced in one still *more* general, whose enunciation might be given in terms, in which the said point should be either *within* or *without*, in *any triangle, the ratio of whose sides is given.*

In any triangle, the *natural sines* of the several angles are proportional to their opposite sides, (see Trigonometry,) and hence, whether the *proportion of the sides*, or the *angles* are given, it is in substance the same thing.

Concerning this problem, the most general view of the case is taken, and the Algebraic solution only is given. The general enunciation of the proposed problem may be presented, thus:

In any plane triangle, having given the proportion of the three sides, and the distance from each angle to a point either within or without the triangle; to find the three sides.

The proposed problem has three conditions:

1st, When the point is within the triangle;
2d, When the point is without the triangle; and
3d, When the point is in one of the sides.

In the 3d condition, the *side* containing said point, is given, for it is necessarily equal to the *sum*, or the *difference* of two of the given distances; and the other two sides may be found by the *given proportion*. Hence, the 1st condition, represented by Fig 1, and the 2d, represented by Fig. 2, remain for discussion.

The two diagrams are so lettered, that the disquisition may be equally applicable to both. Thus:

Let ABC be the given triangle, and O the given point, either within (as in Fig. 1,) or without, (as in Fig. 2.) Also let the lines OA, OB and OC, represent the given distances. Moreover, let C be one of the acute angles, (any triangle has at least *two* acute angles,) included by the two sides AC and BC, and through O let ED be drawn perpendicular to AC, and JK perpendicular to BC; AC and BC being produced, if necessary.

Fig. 1.

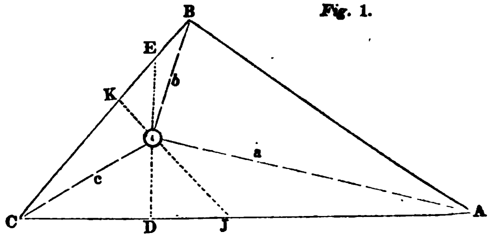
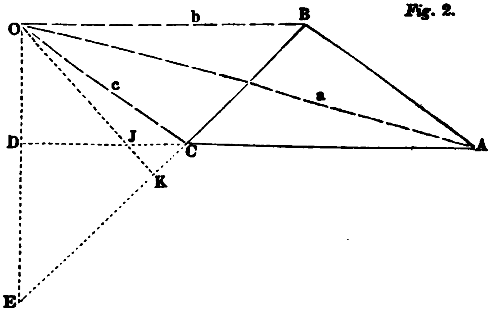


Fig. 2.



Again, let OA be denoted by a; OB by b; OC by c; CD by y; and CK by z. Also let l, m and n be the numbers which denote the given proportion of the three sides, i. e. $AC : AB :: m : n$, and $AC : BC :: m : l$.

Again, let x denote the ratio between the proportional numbers l, m, and n, and the required sides of the triangle to which they are respectively analogous; (for similar triangles have their like sides proportional.) Then $BC = lx$, $AC = mx$, and $AB = nx$.

Having given the three sides l, m, and n, of the triangle, similar to the *required one*, we may by Trigonometry obtain the angles A, B, and C, at pleasure. For instance, taking radius $= 1$, we have

$$\text{Cosine } C = \frac{l^2 + m^2 - n^2}{2lm}$$

And by denoting the secant of the angle C by S, and the tangent by t, we have

$$S = \frac{2lm}{l^2 + m^2 - n^2} \qquad (1)$$

and

$$S^2 = t^2 + 1 \qquad (2)$$

In accordance with the foregoing supposition, in reference to the triangle CKJ, we have (see Trigonometry)

$KJ = t \,.\, z$, and $CJ = S \,.\, z$; consequently,

$JD = \pm CJ \mp CD = \pm S \,.\, z \mp y$.*

By similar triangles, $JK : KC :: JD : DO$

By substitution, $t \,.\, z : z :: \pm S \,.\, z \mp y : DO = \dfrac{\pm S \,.\, z \mp y}{t}$

Again, as $CO^2 = CD^2 + DO^2$, this equation becomes, by substitution and reduction, $t^2c^2 = t^2y^2 + S^2z^2 - 2Szy + y^2$, and by substituting $S^2 - 1$ for t^2, (2) we obtain

$$S^2z^2 - 2Szy + S^2y^2 = t^2c^2 \qquad (3)$$

In the triangle AOC we have, by geometry, $OA^2 = OC^2 + CA^2 \mp 2CA \,.\, CD$*, or by substitution, $a^2 = c^2 + m^2x^2 \mp 2mxy$.

Hence,
$$y = \frac{(a^2 - z) - m^2x^2}{\mp 2mx}, \qquad (4)$$

In like manner, in the triangle COB, $OB^2 = OC^2 + CB^2 \mp 2CB \,.\, CK$; that is, $b^2 = c^2 + l^2x^2 \mp 2lxz$.

Hence,
$$z = \frac{(b^2 - c^2) - l^2x^2}{\mp 2lx}, \qquad (5)$$

Now by combining equations (5), (4) and (3), we have

$$S^2 \times \frac{l^4x^4 - 2l^2(b^2-c^2)x^2 + (b^2-c^2)^2}{4l^2x^2} + S^2 \times \frac{m^4x^4 - 2m^2(a^2-c^2)x^2 + (a^2-c^2)^2}{4m^2x_2}$$

$$-2S \times \frac{l^2m^2x^4 - [l^2(a^2-c^2) + m^2(b^2-c^2)]x^2 + (a^2-c^2)(b^2-c^2)}{4lmx^2} = t^2c^2.$$

By substituting $S^2 - 1$, for t^2, (2) clearing denominators, and reducing, we obtain

$$x^4 - 2 \,.\, \frac{Sl(Sm-l)(a^2-c^2) + Sm(Sl-m)(b^2-c^2) + 2lm(S^2-1)c^2}{Slm(Sl^2 - 2lm + Sm^2)} \times x^2 =$$

$$-\frac{S^2l^2(a^2-c^2)^2 - 2lmS(a^2-c^2)(b^2-c^2) + 2S^2m^2(b^2-c^2)^2}{S^2l^2m^2(Sl^2 - 2lm + Sm^2)}. \qquad (6)$$

---

* The upper signs (of the double signs) are taken when the point D is between A and C, and the point K, between B and C: but when those points fall on those lines produced, the lower signs are taken.

The above equation takes the form of

$$x^4-2px^2=-q, \text{ and } x=\sqrt{p\mp\sqrt{p^2-q}} \qquad (7)$$

By substituting the value of S, (1) in equation (6), and reducing, we obtain for the representatives p and q, in equation (7), the following—thus,

$$p=\frac{(-l^2+m^2+n^2)l^2a^2+(l^2-m^2+n^2)m^2b^2+(l^2+m^2-n^2)n^2c^2}{2l^2m^2n^2}$$

$$\text{and, } p^2-q=\frac{1}{2l^2m^2n^2}\left(\begin{array}{l}(l+m+n)(l+m-n)(l-m+n)\\ \times(-l+m+n)(la+mb+nc)(la+mb-nc)\\ \times(la-mb+nc)(-la+mb+nc)\end{array}\right)$$

Now by substituting in equation (7) the value of p, and p^2-q, as just obtained; and arranging, for convenience sake, the several factors in another form, we have the following

GENERAL FORMULA,

For any triangle whatever.

$$x=\frac{1}{lmn}\cdot\sqrt{\frac{1}{2}\cdot\left(\begin{array}{l}+(-l^2+m^2+n^2)l^2a^2\\ +(l^2-m^2+n^2)m^2b^2\\ +(l^2+m^2-n^2)n^2c^2\end{array}\right)\pm\frac{1}{2}\sqrt{\left(\begin{array}{r}(l+m+n)\\ \times(l+m-n)\\ \times(l-m+n)\\ \times(-l+m+n)\end{array}\right)\times\left(\begin{array}{r}(la+mb+nc)\\ \times(la+mb-nc)\\ \times(la-mb+nc)\\ \times(-la+mb+nc)\end{array}\right)}}$$

Schol. In the above general formula, the positive sign is used when the point is *within* the triangle; the negative sign when the point is *without*.

By denoting the radical part of the above formula by k, we have

$$x=\frac{k}{lmn};\ \text{and } AC=\frac{k}{ln},\ BC=\frac{k}{mn},\ \text{and } AB=\frac{k}{lm}$$

Corol. When the above *general formula* is applied to equilateral triangles, l, m, and n become equal to each other, and each equal to a unit; hence, it takes the form of the following

GENERAL FORMULA,

For any Equilateral Triangle.

$$x=\sqrt{\frac{a^2+b^2+c^2}{2}\pm\frac{1}{2}\sqrt{3(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}}$$

By this last formulæ we solve the required poblem, thus:—

$$x=\sqrt{\frac{(30)^2+(29)^2+(20)^2}{2}+\frac{1}{2}\sqrt{3\left(\begin{array}{c}(30+29+20)(30+29-20)\\ \times(30-29+20)(-30+29+20)\end{array}\right)}}$$

or by reduction,

$$x=\sqrt{\frac{2141}{2}+\frac{1}{2}\sqrt{3\times79\times39\times21\times19}}$$

$x=45{,}063$ rods; one side of the triangle.

The area of said triangle, is 879,28 square rods.

PROBLEM.

In a given triangle *ABC*, the sides *AC*, *AB*, and *BC*, are to each other, as 5, 3 and 7.

Furthermore, from a given point, the distance to the angle *A*, is 10 rods, the distance to the angle *B*, is 21 rods, and that to the angle *C*, is 25 rods.

What are the lengths of the several sides of the triangle,— 1. When the point is in the triangle? and, 2. When the point is out of it?

Now, from the analogy existing between this problem, and that whence the general formula was derived, it is evident, that, in this case, $n=3$, $m=5$, $l=7$, $a=10$, $b=21$, and $c=25$.

Consequently by making the proper substitutions in the general formula, and reducing, we obtain

$$x=\frac{1}{105}\sqrt{327975 \pm 136244{,}265934.}$$

Hence, $x=6,\ 4889$, or $x=4,\ 17019$.

Therefore, when the point is in the triangle, we have

$AC=mx=32{,}4445$, $AB=nx=19{,}4667$, and,
$BC=lx=45{,}4223$ rods.

When the point is out of the triangle, we have

$AC=mx=20{,}85095$, $AB=nx=12{,}51057$, and,
$BC=lx=29{,}19133$ rods.

ERRATA.

Page	12,	8 lines	from	the	top, cancel, "from the first to the $(m+1)$th order, and."
"	13,	3 "	"	"	bottom, for "formula" read *formulæ*.
"	17,	10 "	"	"	top, add after sup. *equal to* $n-(z-1)$.
"	19,	2 "	"	"	bottom, for "a" read **a**—bottom line, for "b" read **b**.
"	22,	3 "	"	"	top, for "any" read *an*.
"	26,	16 "	"	"	top, for "is less" read, *is units less*—bottom line, add after whatever, *as the prim. order*.
"	28,	22 "	"	"	top, commence this line with 60.
"	"	27 "	"	"	top, add after Art. 39, *which is the same, as substituting in the formula* (*Art.* 57,) 1 *for m*.